Vector Spaces and Subspaces

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Vector Space V

It is a **data set** V plus a **toolkit** of eight (8) algebraic properties. The data set consists of packages of data items, called **vectors**, denoted $\vec{X}$, $\vec{Y}$ below.

Closure	The operations $\vec{X} + \vec{Y}$ and $k\vec{X}$ are defined and result in a new vector which is also in the set V.	
Addition	$\vec{X} + \vec{Y} = \vec{Y} + \vec{X}$	commutative
	$\vec{X} + (\vec{Y} + \vec{Z}) = (\vec{Y} + \vec{X}) + \vec{Z}$	associative
	Vector $\vec{0}$ is defined and $\vec{0} + \vec{X} = \vec{X}$	zero
	Vector $-\vec{X}$ is defined and $\vec{X} + (-\vec{X}) = \vec{0}$	negative
Scalar multiply	$k(\vec{X} + \vec{Y}) = k\vec{X} + k\vec{Y}$	distributive I
	$(k_1 + k_2)\vec{X} = k_1\vec{X} + k_2\vec{X}$	distributive II
	$k_1(k_2\vec{X}) = (k_1 k_2)\vec{X}$	distributive III
	$1\vec{X} = \vec{X}$	identity

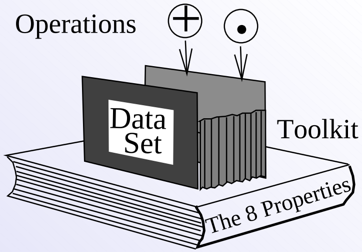

Figure 1. A *Vector Space* is a data set, operations $\boxplus$ and $\boxdot$, and the 8-property toolkit.

Definition of Subspace

A **subspace** $\boldsymbol{S}$ of a vector space $\boldsymbol{V}$ is a nonvoid subset of $\boldsymbol{V}$ which under the operations $+$ and $\cdot$ of $\boldsymbol{V}$ forms a vector space in its own right.

Subspace Criterion

Let $\boldsymbol{S}$ be a subset of $\boldsymbol{V}$ such that

1. Vector $\mathbf{0}$ is in $\boldsymbol{S}$.
2. If $\vec{\boldsymbol{X}}$ and $\vec{\boldsymbol{Y}}$ are in $\boldsymbol{S}$, then $\vec{\boldsymbol{X}} + \vec{\boldsymbol{Y}}$ is in $\boldsymbol{S}$.
3. If $\vec{\boldsymbol{X}}$ is in $\boldsymbol{S}$, then $\boldsymbol{c}\vec{\boldsymbol{X}}$ is in $\boldsymbol{S}$.

Then $\boldsymbol{S}$ is a subspace of $\boldsymbol{V}$.

Items **2**, **3** can be summarized as *all linear combinations of vectors in $\boldsymbol{S}$ are again in $\boldsymbol{S}$*. In proofs using the criterion, items 2 and 3 may be replaced by

$$\boldsymbol{c}_1\vec{\boldsymbol{X}} + \boldsymbol{c}_2\vec{\boldsymbol{Y}} \text{ is in } \boldsymbol{S}.$$

Subspaces are Working Sets

We call a subspace $\boldsymbol{S}$ of a vector space $\boldsymbol{V}$ a **working set**, because the purpose of identifying a subspace is to shrink the original data set $\boldsymbol{V}$ into a smaller data set $\boldsymbol{S}$, customized for the application under study.

A Key Example. Let $\boldsymbol{V}$ be ordinary space $\boldsymbol{R}^{\mathbf{3}}$ and let $\boldsymbol{S}$ be the plane of action of a planar kinematics experiment. The data set for the experiment is all $\mathbf{3}$-vectors $\mathbf{v}$ in $\boldsymbol{V}$ collected by a data recorder. Detected and recorded is the $\mathbf{3}$D position of a particle which has been constrained to a plane. The plane of action $\boldsymbol{S}$ is computed as a homogeneous equation like 2x+3y+1000z=0, the equation of a plane, from the recorded data set in $\boldsymbol{V}$. After least squares is applied to find the optimal equation for $\boldsymbol{S}$, then $\boldsymbol{S}$ replaces the larger data set $\boldsymbol{V}$. The customized smaller set $\boldsymbol{S}$ is the *working set* for the kinematics problem.

The Kernel Theorem

Theorem 1 (Kernel Theorem)
Let V be one of the vector spaces R^n and let A be an $m \times n$ matrix. Define a smaller set S of data items in V by the kernel equation

$$S = \{\mathrm{x} \;:\; \mathrm{x} \text{ in } V, \quad A\mathrm{x} = 0\}.$$

Then S is a subspace of V.
In particular, operations of addition and scalar multiplication applied to data items in S give answers back in S, and the 8-property toolkit applies to data items in S.

Proof: Zero is in V because $A0 = 0$ for any matrix A. To verify the subspace criterion, we verify that $\mathrm{z} = c_1\mathrm{x} + c_2\mathrm{y}$ for x and y in V also belongs to V. The details:

$$\begin{aligned} A\mathrm{z} &= A(c_1\mathrm{x} + c_2\mathrm{y}) \\ &= A(c_1\mathrm{x}) + A(c_2\mathrm{y}) \\ &= c_1A\mathrm{x} + c_2A\mathrm{y} \\ &= c_10 + c_20 && \text{Because } A\mathrm{x} = A\mathrm{y} = 0\text{, due to } \mathrm{x}, \mathrm{y} \text{ in } V. \\ &= 0 && \text{Therefore, } A\mathrm{z} = 0\text{, and } \mathrm{z} \text{ is in } V. \end{aligned}$$

The proof is complete.

Not a Subspace Theorem

Theorem 2 (Testing S not a Subspace)
Let V be an abstract vector space and assume S is a subset of V. Then S is not a subspace of V provided one of the following holds.

(1) The vector $\mathbf{0}$ is not in S.

(2) Some $\mathbf{x}$ and $-\mathbf{x}$ are not both in S.

(3) Vector $\mathbf{x} + \mathbf{y}$ is not in S for some $\mathbf{x}$ and $\mathbf{y}$ in S.

Proof: The theorem is justified from the *Subspace Criterion*.
1. The criterion requires $\mathbf{0}$ is in S.
2. The criterion demands $c\mathbf{x}$ is in S for all scalars c and all vectors $\mathbf{x}$ in S.
3. According to the subspace criterion, the sum of two vectors in S must be in S.

Definition of Independence and Dependence

A list of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ in a vector space V are said to be **independent** provided every linear combination of these vectors is uniquely represented. **Dependent** means **not independent**.

Unique representation

An equation

$$a_1\mathbf{v}_1 + \cdots + a_k\mathbf{v}_k = b_1\mathbf{v}_1 + \cdots + b_k\mathbf{v}_k$$

implies matching coefficients: $a_1 = b_1, \ldots, a_k = b_k$.

Independence Test

Form the system in unknowns $c_1, \ldots, c_k$

$$c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k = \mathbf{0}.$$

Solve for the unknowns [how to do this depends on V]. Then the vectors are independent if and only if the unique solution is $c_1 = c_2 = \cdots = c_k = 0$.

Independence test for two vectors $\mathbf{v}_1$, $\mathbf{v}_2$

In an abstract vector space V, form the equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}.$$

Solve this equation for c_1, c_2.

Then $\mathbf{v}_1$, $\mathbf{v}_2$ are independent in V if and only if the system has unique solution $c_1 = c_2 = 0$.

Geometry and Independence

- One fixed vector is independent if and only if it is nonzero.
- Two fixed vectors are independent if and only if they form the edges of a parallelogram of positive area.
- Three fixed vectors are independent if and only if they are the edges of a parallelepiped of positive volume.

In an abstract vector space $\boldsymbol{V}$, two vectors [two data packages] are independent if and only if one is not a scalar multiple of the other.

Illustration

Vectors $\mathbf{v_1} = \mathbf{cos}\,\boldsymbol{x}$ and $\mathbf{v_2} = \mathbf{sin}\,\boldsymbol{x}$ are two data packages [graphs] in the vector space $\boldsymbol{V}$ of continuous functions. They are independent because one graph is not a scalar multiple of the other graph.

An Illustration of the Independence Test

Two column vectors are tested for independence by forming the system of equations $\boldsymbol{c}_1\mathbf{v}_1 + \boldsymbol{c}_2\mathbf{v}_2 = \mathbf{0}$, e.g,

$$\boldsymbol{c}_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \boldsymbol{c}_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This is a homogeneous system $\boldsymbol{A}\mathbf{c} = \mathbf{0}$ with

$$\boldsymbol{A} = \begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} \boldsymbol{c}_1 \\ \boldsymbol{c}_2 \end{pmatrix}.$$

The system $\boldsymbol{A}\mathbf{c} = \mathbf{0}$ can be solved for $\mathbf{c}$ by frame sequence methods. Because $\mathbf{rref}(\boldsymbol{A}) = \boldsymbol{I}$, then $\boldsymbol{c}_1 = \boldsymbol{c}_2 = 0$, which verifies independence.

If the system $\boldsymbol{A}\mathbf{c} = \mathbf{0}$ is square, then $\mathbf{det}(\boldsymbol{A}) \neq \mathbf{0}$ applies to test independence.

There is **no chance to use determinants** when the system is not square, e.g., consider the homogeneous system

$$\boldsymbol{c}_1 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \boldsymbol{c}_2 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

It has vector-matrix form $\boldsymbol{A}\mathbf{c} = \mathbf{0}$ with $\mathbf{3} \times \mathbf{2}$ matrix $\boldsymbol{A}$, for which $\mathbf{det}(\boldsymbol{A})$ is undefined.

Rank Test

In the vector space $\boldsymbol{R}^n$, the independence test leads to a system of $\boldsymbol{n}$ linear homogeneous equations in $\boldsymbol{k}$ variables $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_k$. The test requires solving a matrix equation $\boldsymbol{A}\mathbf{c} = \mathbf{0}$. The signal for independence is **zero free variables**, or nullity zero, or equivalently, maximal rank. To justify the various statements, we use the relation $\mathbf{nullity}(\boldsymbol{A}) + \mathbf{rank}(\boldsymbol{A}) = \boldsymbol{k}$, where $\boldsymbol{k}$ is the column dimension of $\boldsymbol{A}$.

Theorem 3 (Rank-Nullity Test)
Let $\mathbf{v}_1, \ldots, \mathbf{v}_k$ be $\boldsymbol{k}$ column vectors in $\boldsymbol{R}^n$ and let $\boldsymbol{A}$ be the augmented matrix of these vectors. The vectors are independent if $\mathbf{rank}(\boldsymbol{A}) = \boldsymbol{k}$ and dependent if $\mathbf{rank}(\boldsymbol{A}) < \boldsymbol{k}$. The conditions are equivalent to $\mathbf{nullity}(\boldsymbol{A}) = \mathbf{0}$ and $\mathbf{nullity}(\boldsymbol{A}) > \mathbf{0}$, respectively.

Determinant Test

In the unusual case when the system arising in the independence test can be expressed as $A\mathbf{c} = \mathbf{0}$ and A is square, then $\det(A) = 0$ detects dependence, and $\det(A) \neq 0$ detects independence. The reasoning is based upon the adjugate formula $A^{-1} = \textbf{adj}(A)/\det(A)$, valid exactly when $\det(A) \neq 0$.

Theorem 4 (Determinant Test)
Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be n column vectors in R^n and let A be the augmented matrix of these vectors. The vectors are independent if $\det(A) \neq 0$ and dependent if $\det(A) = 0$.

Sampling Test

Let functions $f_1, \ldots, f_n$ be given and let $x_1, \ldots, x_n$ be distinct x-sample values. Define

$$A = \begin{pmatrix} f_1(x_1) & f_2(x_1) & \cdots & f_n(x_1) \\ f_1(x_2) & f_2(x_2) & \cdots & f_n(x_2) \\ \vdots & \vdots & \cdots & \vdots \\ f_1(x_n) & f_2(x_n) & \cdots & f_n(x_n) \end{pmatrix}.$$

Then $\det(A) \neq 0$ implies $f_1, \ldots, f_n$ are independent functions.

Proof

We'll do the proof for $n = 2$. Details are similar for general n. Assume $c_1 f_1 + c_2 f_2 = 0$. Then for all x, $c_1 f_1(x) + c_2 f_2(x) = 0$. Choose $x = x_1$ and $x = x_2$ in this relation to get $A\mathbf{c} = \mathbf{0}$, where $\mathbf{c}$ has components c_1, c_2. If $\det(A) \neq 0$, then A^{-1} exists, and this in turn implies $\mathbf{c} = A^{-1}A\mathbf{c} = \mathbf{0}$. We conclude f_1, f_2 are independent.

Wronskian Test

Let functions $f_1, \ldots, f_n$ be given and let x_0 be a given point. Define

$$W = \begin{pmatrix} f_1(x_0) & f_2(x_0) & \cdots & f_n(x_0) \\ f_1'(x_0) & f_2'(x_0) & \cdots & f_n'(x_0) \\ \vdots & \vdots & \cdots & \vdots \\ f_1^{(n-1)}(x_0) & f_2^{(n-1)}(x_0) & \cdots & f_n^{(n-1)}(x_0) \end{pmatrix}.$$

Then $\det(W) \neq 0$ implies $f_1, \ldots, f_n$ are independent functions. The matrix W is called the **Wronskian Matrix** of $f_1, \ldots, f_n$ and $\det(W)$ is called the **Wronskian determinant**.

Proof

We'll do the proof for $n = 2$. Details are similar for general n. Assume $c_1 f_1 + c_2 f_2 = 0$. Then for all x, $c_1 f_1(x) + c_2 f_2(x) = 0$ and $c_1 f_1'(x) + c_2 f_2'(x) = 0$. Choose $x = x_0$ in this relation to get $W\mathbf{c} = 0$, where $\mathbf{c}$ has components c_1, c_2. If $\det(W) \neq 0$, then W^{-1} exists, and this in turn implies $\mathbf{c} = W^{-1}W\mathbf{c} = 0$. We conclude f_1, f_2 are independent.

Atoms

Definition. A function is called an **atom** provided it has coefficient $\mathbf{1}$ and is obtained as the real or imaginary part of the expression

$$\boldsymbol{x^k e^{ax}(\cos bx + i \sin bx)}.$$

The powers $\mathbf{1}$, $\boldsymbol{x}$, $\boldsymbol{x^2}$, … are atoms (select $\boldsymbol{a = b = 0}$). Multiples of these powers by $\boldsymbol{\cos bx}$, $\boldsymbol{\sin bx}$ are also atoms. Finally, multiplying all these atoms by $\boldsymbol{e^{ax}}$ expands and completes the list of atoms.

Illustration

We show the powers $\mathbf{1}$, $\boldsymbol{x}$, $\boldsymbol{x^2}$, $\boldsymbol{x^3}$ are independent atoms by applying the Wronskian Test:

$$W = \begin{pmatrix} 1 & x_0 & x_0^2 & x_0^3 \\ 0 & 1 & 2x_0 & 3x_0^2 \\ 0 & 0 & 2 & 6x_0 \\ 0 & 0 & 0 & 6 \end{pmatrix}.$$

Then $\det(W) = 12 \neq 0$ implies the functions $\mathbf{1}$, $\boldsymbol{x}$, $\boldsymbol{x^2}$, $\boldsymbol{x^3}$ are linearly independent.

Subsets of Independent Sets are Independent

Suppose $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ make an independent set and consider the subset $\mathbf{v}_1, \mathbf{v}_2$. If

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}$$

then also

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$$

where $c_3 = 0$. Independence of the larger set implies $c_1 = c_2 = c_3 = 0$, in particular, $c_1 = c_2 = 0$, and then $\mathbf{v}_1, \mathbf{v}_2$ are indpendent.

Theorem 5 (Subsets and Independence)

- A non-void subset of an independent set is also independent.
- Non-void subsets of dependent sets can be independent or dependent.

Atoms and Independence

Theorem 6 (Independence of Atoms)
Any list of distinct atoms is linearly independent.

Unique Representation

The theorem is used to extract equations from relations involving atoms. For instance:

$$(c_1 - c_2)\cos x + (c_1 + c_3)\sin x + c_1 + c_2 = 2\cos x + 5$$

implies

$$\begin{aligned} c_1 - c_2 &= 2, \\ c_1 + c_3 &= 0, \\ c_1 + c_2 &= 5. \end{aligned}$$

Atoms and Differential Equations

It is known that solutions of linear constant coefficient differential equations of order n and also systems of linear differential equations with constant coefficients have a general solution which is a linear combination of atoms.

- The harmonic oscillator $y'' + b^2 y = 0$ has general solution $y(x) = c_1 \cos bx + c_2 \sin bx$. This is a linear combination of the two atoms $\cos bx$, $\sin bx$.
- The third order equation $y''' + y' = 0$ has general solution $y(x) = c_1 \cos x + c_2 \sin x + c_3$. The solution is a linear combination of the independent atoms $\cos x$, $\sin x$, 1.
- The linear dynamical system $x'(t) = y(t)$, $y'(t) = -x(t)$ has general solution $x(t) = c_1 \cos t + c_2 \sin t$, $y(t) = -c_1 \sin t + c_2 \cos t$, each of which is a linear combination of the independent atoms $\cos t$, $\sin t$.