

## Elementary Matrices and Frame Sequences

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## Elementary Matrices

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**Definition.** An elementary matrix  $E$  is the result of applying a combination, multiply or swap rule to the identity matrix.

An elementary matrix is then the **second frame** after a `combo`, `swap` or `mult` toolkit operation which has been applied to a **first frame** equal to the identity matrix.

**Example:**

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

First frame = identity matrix.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{pmatrix}$$

Second frame  
Elementary combo matrix  
`combo(1, 3, -5)`

## Computer algebra systems and elementary matrices

The computer algebra system maple displays typical  $4 \times 4$  elementary matrices (C=Combination, M=Multiply, S=Swap) as follows.

```
with(linalg):          with(LinearAlgebra):
Id:=diag(1,1,1,1);    Id:=IdentityMatrix(4);
C:=addrow(Id,2,3,c);  C:=RowOperation(Id,[3,2],c);
M:=mulrow(Id,3,m);    M:=RowOperation(Id,3,m);
S:=swaprow(Id,1,4);   S:=RowOperation(Id,[4,1]);
```

The answers:

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & c & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & m & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$S = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

## Constructing elementary matrices $E$ and their inverses $E^{-1}$ \_\_\_\_\_

**Mult** Change a one in the identity matrix to symbol  $m \neq 0$ .

**Combo** Change a zero in the identity matrix to symbol  $c$ .

**Swap** Interchange two rows of the identity matrix.

### Constructing $E^{-1}$ from elementary matrix $E$

**Mult** Change diagonal multiplier  $m \neq 0$  in  $E$  to  $1/m$ .

**Combo** Change multiplier  $c$  in  $E$  to  $-c$ .

**Swap** The inverse of  $E$  is  $E$  itself.

## Fundamental Theorem on Elementary Matrices

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### Theorem 1 (Frame sequences and elementary matrices)

In a frame sequence, let the second frame  $A_2$  be obtained from the first frame  $A_1$  by a `combo`, `swap` or `mult` toolkit operation. Let  $n$  equal the row dimension of  $A_1$ . Then there is correspondingly an  $n \times n$  `combo`, `swap` or `mult` elementary matrix  $E$  such that

$$A_2 = EA_1.$$

### Theorem 2 (The `rref` and elementary matrices)

Let  $A$  be a given matrix of row dimension  $n$ . Then there exist  $n \times n$  elementary matrices  $E_1, E_2, \dots, E_k$  such that

$$\text{rref}(A) = E_k \cdots E_2 E_1 A.$$

### Proof of Theorem 1

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The first result is the observation that left multiplication of matrix  $A_1$  by elementary matrix  $E$  gives the answer  $A_2 = EA_1$  which is obtained by applying the corresponding `combo`, `swap` or `mult` toolkit operation. This fact is discovered by doing examples, then a formal proof can be constructed (not presented here).

### Proof of Theorem 2

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The second result applies the first result multiple times to obtain elementary matrices  $E_1, E_2, \dots$  which represent the multiply, combination and swap operations performed in the frame sequence which take the First Frame  $A_1 = A$  into the Last Frame  $A_{k+1} = \text{rref}(A_1)$ . Combining the identities

$$A_2 = E_1 A_1, \quad A_3 = E_2 A_2, \quad \dots, \quad A_{k+1} = E_k A_k$$

gives the matrix multiply equation

$$A_{k+1} = E_k E_{k-1} \cdots E_2 E_1 A_1$$

or equivalently the theorem's result, because  $A_{k+1} = \text{rref}(A)$  and  $A_1 = A$ .

## A certain 6-frame sequence \_\_\_\_\_.

$$A_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 0 \\ 3 & 6 & 3 \end{pmatrix} \quad \text{Frame 1, original matrix.}$$

$$A_2 = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & -6 \\ 3 & 6 & 3 \end{pmatrix} \quad \text{Frame 2, combo(1,2,-2).}$$

$$A_3 = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 3 & 6 & 3 \end{pmatrix} \quad \text{Frame 3, mult(2,-1/6).}$$

$$A_4 = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & -6 \end{pmatrix} \quad \text{Frame 4, combo(1,3,-3).}$$

$$A_5 = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{Frame 5, combo(2,3,-6).}$$

$$A_6 = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{Frame 6, combo(2,1,-3). Found rref}(A_1).$$

## Continued

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The corresponding  $3 \times 3$  elementary matrices are

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{Frame 2, combo}(1,2,-2) \text{ applied to } I.$$

$$E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1/6 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{Frame 3, mult}(2,-1/6) \text{ applied to } I.$$

$$E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix} \quad \text{Frame 4, combo}(1,3,-3) \text{ applied to } I.$$

$$E_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -6 & 1 \end{pmatrix} \quad \text{Frame 5, combo}(2,3,-6) \text{ applied to } I.$$

$$E_5 = \begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{Frame 6, combo}(2,1,-3) \text{ applied to } I.$$



## Frame Sequence Details

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$$A_2 = E_1 A_1$$

Frame 2,  $E_1$  equals  
combo(1,2,-2) on  $I$ .

$$A_3 = E_2 A_2$$

Frame 3,  $E_2$  equals  
mult(2,-1/6) on  $I$ .

$$A_4 = E_3 A_3$$

Frame 4,  $E_3$  equals  
combo(1,3,-3) on  $I$ .

$$A_5 = E_4 A_4$$

Frame 5,  $E_4$  equals  
combo(2,3,-6) on  $I$ .

$$A_6 = E_5 A_5$$

Frame 6,  $E_5$  equals  
combo(2,1,-3) on  $I$ .

$$A_6 = E_5 E_4 E_3 E_2 E_1 A_1$$

Summary frames 1-6.

Then

$$\text{rref}(A_1) = E_5 E_4 E_3 E_2 E_1 A_1,$$

which is the result of the Theorem.

## Fundamental Theorem Illustrated

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The summary:

$$A_6 = \begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -6 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{6} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A_1$$

Because  $A_6 = \text{rref}(A_1)$ , the above equation gives the inverse relationship

$$A_1 = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} E_5^{-1} \text{rref}(A_1).$$

Each inverse matrix is simplified by the rules for constructing  $E^{-1}$  from elementary matrix  $E$ , the result being

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 6 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{rref}(A_1)$$

### Theorem 3 (RREF Inverse Method)

$$\text{rref}(\text{aug}(A, I)) = \text{aug}(I, B) \quad \text{if and only if} \quad AB = I.$$

**Proof:** For *any* matrix  $E$  there is the matrix multiply identity

$$E \text{ aug}(C, D) = \text{aug}(EC, ED).$$

This identity is proved by arguing that each side has identical columns. For example,  $\text{col}(\text{LHS}, 1) = E \text{ col}(C, 1) = \text{col}(\text{RHS}, 1)$ .

Assume  $C = \text{aug}(A, I)$  satisfies  $\text{rref}(C) = \text{aug}(I, B)$ . The fundamental theorem of elementary matrices implies  $E_k \cdots E_1 C = \text{rref}(C)$ . Then

$$\text{rref}(C) = \text{aug}(E_k \cdots E_1 A, E_k \cdots E_1 I) = \text{aug}(I, B)$$

implies that  $E_k \cdots E_1 A = I$  and  $E_k \cdots E_1 I = B$ . Together,  $AB = I$  and then  $B$  is the inverse of  $A$ .

Conversely, assume that  $AB = I$ . Then  $A$  has inverse  $B$ . The fundamental theorem of elementary matrices implies the identity  $E_k \cdots E_1 A = \text{rref}(A) = I$ . It follows that  $B = E_k \cdots E_1$ . Then  $\text{rref}(C) = E_k \cdots E_1 \text{aug}(A, I) = \text{aug}(E_k \cdots E_1 A, E_k \cdots E_1 I) = \text{aug}(I, B)$ .

