Examples: Solving nth Order Equations

- Atoms and Base Atoms
- L. Euler's Theorem
- The Atom List
- First Order. Solve $2y' + 5y = 0$.
- Second Order. Solve the equations
  $y'' + 2y' + y = 0$, $y'' + 3y' + 2y = 0$ and $y'' + 2y' + 5y = 0$.
- Third Order. Solve $y''' - y' = 0$ and $y''' - y'' = 0$.
- Fourth Order. Solve $y^{(4)} - y^{(2)} = 0$.

Atoms

Assume symbols a and b are real constants, with $a \neq 0$ and $b > 0$.
A **base atom** is one of the functions

$$1, \quad e^{ax}, \quad \cos bx, \quad \sin bx, \quad e^{ax}\cos bx, \quad e^{ax}\ \sin bx.$$

An **atom** is a base atom multiplied by a power x^m, where $m \geq 0$ is an integer.

Examples

Atoms: 1, $\cos(3x/2)$, x^3, $x\cos(2x)$, $x^2e^{2x}\sin(3x)$, $\cos(\pi x)$, e^{e^2x}

Not an atom: 2, $-x$, $x^{3/2}$, $\tan x$, $\sin(x^2)$, e^{x^2}, $\dfrac{x}{1+x^2}$, $\ln|x|$, $\sinh x$

Euler's Theorem

Theorem 1 (L. Euler)
The function $y = x^m e^{r_1 x}$ is a solution of a constant-coefficient linear homogeneous differential equation of the nth order if and only if $(r - r_1)^{m+1}$ divides the characteristic polynomial.

Euler's theorem is used to construct solutions of the nth order differential equation. The solutions so constructed are n distinct atoms, hence independent. Picard's theorem implies the list of atoms is a basis for the solution space.

Euler's Theorem in Words

To construct solutions of homogeneous constant-coefficient differential equations, use Euler's Theorem as follows.

- Find the roots of the characteristic equation.
  - For each real root r, the exponential solution e^{rx} is a base atom solution.
  - For each complex conjugate pair of roots $a \pm bi$, $b > 0$, the functions $e^{ax}\cos bx$, $e^{ax}\sin bx$ are base atom solutions.
- Multiply the base atoms by the powers 1, x, x^2, $\ldots$, x^{k-1}, where k is the multiplicity of the root. The atoms so found are all solutions of the differential equation.

The Atom List

1. If r_1 is a real root, then the base atom for r_1 is $e^{r_1 x}$. The root r_1 has multiplicity k provided factor $(r - r_1)^k$ divides the characteristic polynomial, but factor $(r - r_1)^{k+1}$ does not. Multiply the base atom by powers 1, x, …, x^{k-1} (a total of k terms) to obtain the atom list

$$e^{r_1 x}, x e^{r_1 x}, \ldots, x^{k-1} e^{r_1 x}.$$

2. If $r_1 = a + ib$ ($b > 0$ assumed) and its conjugate $r_2 = a - ib$ are roots of the characteristic equation, then the base atoms for this pair of roots (both r_1 and r_2 counted) are

$$e^{ax}\cos bx, \quad e^{ax}\sin bx.$$

For a root of multiplicity k, these real atoms are multiplied by atoms 1, …, x^{k-1} to obtain the complete list of $2k$ atoms

$$\begin{array}{llll} e^{ax}\cos bx, & xe^{ax}\cos bx, & \ldots, & x^{k-1}e^{ax}\cos bx, \\ e^{ax}\sin bx, & xe^{ax}\sin bx, & \ldots, & x^{k-1}e^{ax}\sin bx. \end{array}$$

1 Example (First Order) Solve $2y' + 5y = 0$ by Euler's method, verifying $y_h = c_1 e^{-5x/2}$.

Solution

$2y' + 5y = 0$	Given differential equation.
$2r + 5 = 0$	Characteristic equation. Use the shortcut $y^{(n)} \longrightarrow r^n$.
$r = -5/2$	Exactly one real root.
Atom = $e^{-5x/2}$	For a real root r, the base atom is e^{rx}.
$y_h = c_1 e^{-5x/2}$	The general solution y_h is written by multiplying the atom list by constant c_1.

2 Example (Second Order I) Solve $y'' + 2y' + y = 0$ by Euler's method, showing that $y_h = c_1 e^{-x} + c_2 x e^{-x}$.

Solution

$y'' + 2y' + y = 0$	Given differential equation.
$r^2 + 2r + 1 = 0$	Characteristic equation. Use shortcut $y^{(n)} \longrightarrow r^n$.
$r = -1, -1$	Exactly two real roots.
Base Atom = e^{-x}	For a real root r_1, the base atom is $e^{r_1 x}$.
Atoms = e^{-x}, xe^{-x}	For a root of multiplicity 2, multiply the base atom by powers 1, x, to create 2 atoms.
$y_h = c_1 e^{-x} + c_2 x e^{-x}$	The general solution y_h is written by multiplying the atom list by constants c_1, c_2.

3 Example (Second Order II) Solve $y'' + 3y' + 2y = 0$ by Euler's method, showing $y_h = c_1 e^{-x} + c_2 e^{-2x}$.

Solution

$y'' + 3y' + 2y = 0$	Given differential equation.
$r^2 + 3r + 2 = 0$	Characteristic equation. Use shortcut $y^{(n)} \to r^n$.
$r = -1, -2$	Factorization $(r+2)(r+1) = 0$.
Base Atoms = e^{-x}, e^{-2x}	For a real root r_1, the base atom is $e^{r_1 x}$.
Atoms = $e^{-x},\ e^{-2x}$	For a root of multiplicity 1, multiply the base atom by 1, to create one term. The number of terms created from one base atom always equals the multiplicity of the root.
$y_h = c_1 e^{-x} + c_2 e^{-2x}$	The general solution y_h is written by multiplying the atom list by constants c_1, c_2.

4 Example (Second Order III) Solve $y''+2y'+5y=0$ by Euler's method, showing $y_h = c_1e^{-x}\cos 2x + c_2xe^{-x}\sin 2x$.

Solution

$y'' + 2y' + 5y = 0$	Given differential equation.
$r^2 + 2r + 5 = 0$	Characteristic equation. Shortcut $y^{(n)} \longrightarrow r^n$ used.
$r = -1 + 2i, -1 - 2i$	Factorization $(r+1)^2+4=0$ implies $r+1=\pm 2\sqrt{-1}=\pm 2i$.
Base Atoms = $e^{-x}\cos 2x$, $e^{-x}\sin 2x$	For complex conjugate roots $r = a \pm ib$, the base atoms are $e^{ax}\cos bx$ and $e^{ax}\sin bx$.
Atoms = $e^{-x}\cos 2x,\ e^{-x}\sin 2x$	Multiply the base atoms by 1, because the multiplicity of root $1+2i$ is 1. In general, multiply by 1, x, ..., x^{k-1}, where k is the root multiplicity.
$y_h = c_1e^{-x}\cos 2x + c_2e^{-x}\sin 2x$	Multiply the atom list by constants c_1, c_2. This is the general solution.

5 Example (Third Order I) Solve $y''' - y' = 0$ by Euler's method, showing $y_h = c_1 + c_2 e^{x} + c_3 e^{-x}$.

Solution

$y''' - y' = 0$	Given differential equation.
$r^3 - r = 0$	Characteristic equation. Use $y^{(n)} \longrightarrow r^n$.
$r = 0, 1, -1$	Factorization $r(r+1)(r-1) = 0$.
Base Atoms = $1, e^{-x}, e^{x}$	For a real root r_1, the base atom is $e^{r_1 x}$.
Atoms = $1, e^{-x}, e^{x}$	Each root has multiplicity 1. Multiply each base atom by 1. Generally, multiply by $1, x, \ldots, x^{k-1}$, where k is the root multiplicity.
$y_h = c_1 + c_2 e^{-x} + c_3 e^{x}$	The general solution y_h is written by multiplying the atom list by constants c_1, c_2, c_3.

6 Example (Third Order II) Solve $y''' - y'' = 0$ by Euler's method, showing $y_h = c_1 + c_2 x + c_3 e^x$.

Solution

$y''' - y'' = 0$	Given differential equation.
$r^3 - r^2 = 0$	Characteristic equation. Use $y^{(n)} \longrightarrow r^n$.
$r = 0, 0, 1$	Factorization $r^2(r-1) = 0$.
Base Atoms = $1, e^x$	For a real root r_1, the base atom is $e^{r_1 x}$. Then the base atoms are e^{0x}, e^{1x}, written as 1, e^x.
Atoms = $1, x, e^x$	Because $r = 0$ has multiplicity 2, then multiply base atom 1 (=e^{0x}) by powers 1, x. Root $r = 1$ with multiplicity 1 implies base atom e^x is multiplied by 1. The total number of atoms created is $2 + 1 = 3$, which is the sum of the root multiplicities.
$y_h = c_1 + c_2 x + c_3 e^x$	The general solution y_h is written by multiplying the atom list by constants c_1, c_2, c_3.

7 Example (Fourth Order) Solve $y^{(4)} - y^{(2)} = 0$ by Euler's method, showing that the general solution is $y_h = c_1 + c_2 x + c_3 e^x + c_4 e^{-x}$.

Solution

$y^{(4)} - y^{(2)} = 0$	Given differential equation.
$r^4 - r^2 = 0$	Characteristic equation. Use $y^{(n)} \to r^n$.
$r = 0, 0, 1, -1$	Factorization $r^2(r-1)(r+1) = 0$.
Base Atoms = $1, e^x, e^{-x}$	For a real root r_1, the base atom is $e^{r_1 x}$. The base atoms e^{0x}, e^{1x}, e^{-1x} are written 1, e^x, e^{-x}.
Atoms = $1, x, e^x, e^{-x}$	Multiply the multiplicity 2 base atom 1 by powers 1, x and each of the multiplicity 1 base atoms e^x, e^{-x} by 1. The total number of atoms created is $2 + 1 + 1 = 4$, which is the sum of the root multiplicities.
$y_h = c_1 + c_2 x + c_3 e^x + c_4 e^{-x}$	Multiply the atom list by constants c_1, c_2, c_3, c_4 and add. This is the general solution.