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Top Scores 1 → 4

Differential Equations 2280
Final Exam
Wednesday, 6 May 2009, 8:00-10:00am

Instructions: This in-class exam is 120 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%.

1. (ch3)

100

(1a) [20%] Solve $2v'(t) = -8 + \dfrac{2}{2t+1}v(t)$, $v(0) = -4$. Show all integrating factor steps.

(1b) [10%] Solve for the general solution: $y'' + 4y' + 6y = 0$.

(1c) [10%] Solve for the general solution of the homogeneous constant-coefficient differential equation whose characteristic equation is $r(r^2+r)^2(r^2+9)^2 = 0$.

(1d) [20%] Find a linear homogeneous constant coefficient differential equation of lowest order which has a particular solution $y = x + \sin\sqrt{2}x + e^{-x}\cos 3x$.

(1e) [15%] A particular solution of the equation $mx'' + cx' + kx = F_0\cos(2t)$ happens to be $x(t) = 11\cos 2t + e^{-t}\sin\sqrt{11}t - \sqrt{11}\sin 2t$. Assume m, c, k all positive. Find the unique periodic steady-state solution x_{SS}.

(1f) [25%] Determine for $y''' + y'' = 100 + 2e^{-x} + \sin x$ the shortest trial solution for y_p according to the method of undetermined coefficients. Do not evaluate the undetermined coefficients!

1a) $2v' - \frac{2}{2t+1}v = -8$, $v(0) = -4$

$v' - \frac{1}{2t+1}v = -4$

$\rho = e^{\int -\frac{1}{2t+1}dt} = e^{-\frac{1}{2}\ln(2t+1)} = (2t+1)^{-1/2}$

$\int [(2t+1)^{-1/2}v]' = \int -4(2t+1)^{-1/2}$

$(2t+1)^{-1/2}v = -4\cdot 2\cdot\frac{1}{2}(2t+1)^{1/2} + C$

$(2t+1)^{-1/2}v = -4(2t+1)^{1/2} + C$

$v(t) = -4(2t+1) + \frac{C}{(2t+1)^{1/2}}$

$-4 = -4(1) + C \rightarrow C = 0$

$\boxed{v(t) = -4(2t+1)}$

1b) $y'' + 4y' + 6y = 0$

$r^2 + 4r + 6 = 0$

$r^2 + 4r + 4 + 2 = 0$

$(r+2)^2 + 2 = 0 \rightarrow (r+2)^2 = -2$

$r + 2 = \pm\sqrt{2}i \rightarrow r = -2 \pm \sqrt{2}i$

$\boxed{y = c_1e^{-2t}\cos\sqrt{2}t + c_2e^{-2t}\sin\sqrt{2}t}$

1c) $r(r^2+r)^2(r^2+9)^2 = 0$

$r(r(r+1))^2(r^2+9)^2 = 0$

Roots $= 0, 0, 0, -1, -1, \pm 3i, \pm 3i$

$\boxed{y = c_1 + c_2t + c_3t^2 + c_4e^{-t} + c_5te^{-t} + c_6\cos 3t + c_7\sin 3t + c_8t\cos t + c_9t\sin t}$

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1d) $y = x + \sin\sqrt{2}x + e^{-x}\cos 3x$

Roots $= 0, 0, \pm\sqrt{2}i, -1 \pm 3i$

$r^2(r^2+2)((r+1)^2+9) = 0$

$(r^4 + 2r^2)(r^2 + 2r + 10) = 0$

$r^6 + 2r^5 + 10r^4 + 2r^4 + 4r^3 + 10r^2 = 0$

$r^6 + 2r^5 + 12r^4 + 4r^3 + 20r^2 = 0$

$\boxed{y^{(6)} + 2y^{(5)} + 12y^{(4)} + 4y''' + 20y'' = 0}$

e) $mx'' + cx' + kx = \ldots \cos(2t)$

$x(t) = 11\cos 2t + e^{-t}\sin\sqrt{11}t - \sqrt{11}\sin 2t$

$\cancel{x'(t) = 22\cos 2t + \sqrt{11}e^{-t}\cos\sqrt{11}t - e^{-t}\sin\sqrt{11}t - 2\sqrt{11}\cos 2t}$

$\cancel{x''(t) = -44\sin 2t - 11e^{-t}\sin\sqrt{11}t - \sqrt{11}e^{-t}\cos\sqrt{11}t - \sqrt{11}e^{-t}\cos\sqrt{11}t + e^{-t}\sin\sqrt{11}t + 4\sqrt{11}\sin 2t}$

$\cancel{x''(t) = (4\sqrt{11} - 44)\sin 2t - 10e^{-t}\sin\sqrt{11}t - 2\sqrt{11}e^{-t}\cos\sqrt{11}t}$

$x_{ss}(t) = 11\cos 2t + \cancel{e^{-t}\sin\sqrt{11}t} - \sqrt{11}\sin 2t$

$x_{ss} = 11\cos 2t - \sqrt{11}\sin 2t$

As $t \to \infty$, $e^{-t}\sin\sqrt{11}t \to 0$, so it is not part of the periodic steady-state soluti

As $t \to \infty$, $11\cos 2t$ and $-\sqrt{11}\sin 2t$ are 2π-periodic.

$\boxed{x_{ss} = 11\cos 2t - \sqrt{11}\sin 2t}$

f) $y''' + y'' = 100 + 2e^{-x} + \sin x$

$r^3 + r^2 = 0 \to r^2(r+1) = 0 \to r = 0, 0, -1$

$y_h = c_1 + c_2 x + c_3 e^{-x}$

$g(x) = x^3 f(x) = 100x^3 + 2x^3 e^{-x} + x^3 \sin x$

$\cancel{1} + \cancel{x} + x^2 + \cancel{x^3}$

$\cancel{e^{-x}} + xe^{-x} + \cancel{x^2 e^{-x}} + \cancel{x^3 e^{-x}}$

$\sin x + \cancel{x\sin x} + \cancel{x^2\sin x} + \cancel{x^3\sin x}$

$\cos x + \cancel{x\cos x} + \cancel{x^2\cos x} + \cancel{x^3\cos x}$

$\boxed{y_p = d_1 x^2 + d_2 x e^{-x} + d_3 \sin x + d_4 \cos x}$

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2. (ch5)

(2a) [25%] A certain 2×2 matrix A has eigenpairs $\left(4, \begin{pmatrix} -1 \\ 1 \end{pmatrix}\right), \left(6, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$. The general solution of $\mathbf{x}'(t) = A\mathbf{x}(t)$ is $\mathbf{x}(t) = e^{At}\mathbf{x}(0)$. Eigenanalysis gives a different formula for $\mathbf{x}(t)$. Combine these facts to find e^{At}.

(2b) [25%] Let $A = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}$. The Cayley-Hamilton method says that the solution of $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is given by $\mathbf{x}(t) = e^{3t}\mathbf{c}_1 + e^{5t}\mathbf{c}_2$. Find the constant vectors $\mathbf{c}_1$, $\mathbf{c}_2$ which produce the solution with $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

(2c) [20%] Let $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$. Find the general solution of $\mathbf{x}' = A\mathbf{x}$ using eigenanalysis.

(2d) [20%] Let $A = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$. Find the general solution of $\mathbf{x}' = A\mathbf{x}$ using the exponential matrix. The Laplace resolvent formula implies $\mathcal{L}(e^{At}) = (sI - A)^{-1}$. Putzer's formula also applies. Use any method available to find e^{At}.

(2e) [10%] Let $e^{At} = \begin{pmatrix} e^t & 0 \\ te^t & e^t \end{pmatrix}$. Find A.

2b) $x(t) = e^{3t}\vec{c_1} + e^{5t}\vec{c_2}$

$x' = Ax = 3e^{3t}\vec{c_1} + 5e^{5t}\vec{c_2}$

$x_0 = \vec{c_1} + \vec{c_2}$

$Ax_0 = 3\vec{c_1} + 5\vec{c_2}$

$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \vec{c_1} + \vec{c_2} \rightarrow \vec{c_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \vec{c_2}$

$\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \end{bmatrix} = 3\vec{c_1} + 5\vec{c_2}$

$\begin{bmatrix} 6 \\ 9 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} - 3\vec{c_2} + 5\vec{c_2}$

$\begin{bmatrix} 3 \\ 3 \end{bmatrix} = 2\vec{c_2} \rightarrow \boxed{\vec{c_2} = \begin{bmatrix} 3/2 \\ 3/2 \end{bmatrix}}$

$\vec{c_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 3/2 \\ 3/2 \end{bmatrix} \rightarrow \boxed{\vec{c_1} = \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}}$

2c) $\begin{bmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix} \rightarrow (3-\lambda)(3-\lambda) - 1 = 0$

$\lambda^2 - 6\lambda + 8 = 0$

$(\lambda-4)(\lambda-2) = 0 \rightarrow \lambda_1 = 4, \lambda_2 = 2$

$[A-4I]v = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{matrix} a=1 \\ b=1 \end{matrix} \rightarrow v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$[A-2I]v = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{matrix} a=1 \\ b=-1 \end{matrix} \rightarrow v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\boxed{x = c_1 e^{4t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}}$

2d) $e^{At} = e^{\lambda t}I + te^{\lambda t}(A - \lambda I)$

$e^{At} = e^{3t}I + te^{3t}(A - 3I)$

$e^{At} = \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{3t} \end{bmatrix} + \begin{bmatrix} 0 & te^{3t} \\ 0 & 0 \end{bmatrix}$

$e^{At} = \begin{bmatrix} e^{3t} & te^{3t} \\ 0 & e^{3t} \end{bmatrix}$, $x(0) = \begin{bmatrix} a \\ b \end{bmatrix}$

$x = e^{At}x(0)$

$x = \begin{bmatrix} e^{3t} & te^{3t} \\ 0 & e^{3t} \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \boxed{\begin{bmatrix} ae^{3t} + bte^{3t} \\ be^{3t} \end{bmatrix}}$

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2e) $e^{At} = e^{\lambda t}I + te^{\lambda t}(A - \lambda I)$

$\begin{bmatrix} e^t & 0 \\ te^t & e^t \end{bmatrix} = \begin{bmatrix} e^t & 0 \\ 0 & e^t \end{bmatrix} + te^t[A - I]$

$\begin{bmatrix} 0 & 0 \\ te^t & 0 \end{bmatrix} = te^t(A - I) \rightarrow \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = A - I \rightarrow A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + I \rightarrow A = \boxed{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}}$

2a) $\lambda_1 = 4,\ v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\lambda_2 = 6,\ v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$A = \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 5-\lambda & 1 \\ 1 & 5-\lambda \end{bmatrix} \rightarrow (5-\lambda)(5-\lambda)-1 = 0$$

$$\lambda^2 - 10\lambda + 24 = 0 \rightarrow (\lambda-6)(\lambda-4)=0 \rightarrow \lambda_1 = 4,\ \lambda_2 = 6 \checkmark$$

$$[A-4I]v = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{matrix} a = -1 \\ b = 1 \end{matrix} \rightarrow v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \checkmark$$

$$\rightarrow A = \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix} \checkmark$$

$$[A-6I]v = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{matrix} a = 1 \\ b = 1 \end{matrix} \rightarrow v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \checkmark$$

$$e^{At} = e^{\lambda_1 t} I - \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}(A - \lambda_1 I)$$

$$e^{At} = \begin{bmatrix} e^{4t} & 0 \\ 0 & e^{4t} \end{bmatrix} + \frac{e^{4t} - e^{6t}}{4-6}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} e^{4t} & 0 \\ 0 & e^{4t} \end{bmatrix} + \frac{1}{2}\begin{bmatrix} e^{6t} - e^{4t} & e^{6t} - e^{4t} \\ e^{6t} - e^{4t} & e^{6t} - e^{4t} \end{bmatrix}$$

$$e^{At} = \boxed{\frac{1}{2}\begin{bmatrix} e^{6t} + e^{4t} & e^{6t} - e^{4t} \\ e^{6t} - e^{4t} & e^{6t} + e^{4t} \end{bmatrix}}$$

an educated guess on A ??

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3. (ch6)

✓(3a) [20%] Compute all equilibrium points of $x' = 14x - x^2/2 - xy$, $y' = 16y - y^2/2 - xy$.

✓(3b) [40%] For the nonlinear competition system $x' = 14x - x^2/2 - xy$, $y' = 16y - y^2/2 - xy$, compute the linearization at $(12, 8)$, which is a linear equation $\mathbf{u}' = A\mathbf{u}$. Classify the unique equilibrium $(0, 0)$ of $\mathbf{u}' = A\mathbf{u}$ as a node, spiral, center, saddle. What classification can be deduced for the nonlinear system?

✓(3c) [20%] Write the nonlinear pendulum equation $x'' + \sin x = 0$ as a nonlinear dynamical system. Then find all equilibrium points of the dynamical system.

-4 ✓(3d) [20%] The dynamical system $x' = y + 1$, $y' = x + y$ can be solved by various methods. Solve it and classify its unique equilibrium point $x = 1$, $y = -1$ as a node, saddle, center or spiral.

(3a) $x' = 14x - \frac{x^2}{2} - xy \qquad x' = x\left(-\frac{x}{2} + 14 - y\right)$ • $(0,0)$

$y' = 16y - \frac{y^2}{2} - xy \qquad y' = y\left(-\frac{y}{2} + 16 - x\right)$ • $(0,32)$ • $(28,0)$

$-\frac{x}{2} + 14 - y = 0 \qquad -\frac{x}{2} - y = -14$

$-\frac{y}{2} + 16 - x = 0 \qquad -x - \frac{y}{2} = -16$

$x + 2y = 28$

$x + \frac{y}{2} = 16$

$\frac{3}{2} y = 12$

$y = 8$

$x = 12$

• $(12,8)$

equilibrium points: $(0,0)$, $(0,32)$, $(28,0)$, $(12,8)$ ✓

(3b) $J = \begin{pmatrix} \frac{\partial}{\partial x} f & \frac{\partial}{\partial y} f \end{pmatrix}$

$$J = \begin{pmatrix} 14 - x - y & -x \\ -y & 16 - y - x \end{pmatrix}$$

$$J(12,8) = \begin{pmatrix} -6 & -12 \\ -8 & -4 \end{pmatrix}$$

linearization at $(12,8)$ $\vec{u}' = \begin{pmatrix} -6 & -12 \\ -8 & -4 \end{pmatrix} \vec{u}$

$(-6-\lambda)(-4-\lambda) + 96 = 0$

$24 - 10\lambda - \lambda^2 + 96 = 0$

$\lambda^2 + 10\lambda - 120 = 0$

$(\lambda + 40)(\lambda - 30) = 0$ atoms e^{40t}, e^{-36t} ✓ it is a saddle because unbounded at $t \to \infty$ and $t \to -\infty$

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→

David Daruais

(ch 5)

(2c) $\begin{pmatrix} e^t & 0 \\ te^t & e^t \end{pmatrix} = e^{\lambda_1 t} I + te^{\lambda_1 t}(A - \lambda_1 I)$

$= e^t I + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} te^t \quad \lambda = 1 \quad A - I = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$\boxed{A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}}$ ✓

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3. (ch 6)

(3c) $x_1 = x$ $\quad x_2' + \sin x_1 = 0$

$x_2 = x' = x_1'$ $\quad \begin{cases} x_1' = x_2 \\ x_2' = -\sin x_1 \end{cases}$ ✓

$x_2' = x''$

equilibrium points

$\boxed{x_1 = k\pi,\ x_2 = 0 \text{ for all } k}$

(3d) $x' = y + 1$ $\quad u' = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \vec{u} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad \lambda(\lambda - 1) - 1 = 0$

$y' = x + y$ $\quad \lambda^2 - \lambda - 1 = 0$

$\mathcal{L}(1) = 1/s$

$\mathcal{L}(t) = 1/s^2$

$s\mathcal{L}(u) = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \mathcal{L}(u) + \mathcal{L}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + u(0)$

$\mathcal{L}(u)\begin{pmatrix} s & 1 \\ 1 & s-1 \end{pmatrix} = \mathcal{L}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + u(0) = \begin{pmatrix} 1/s \\ 0 \end{pmatrix} + u_0$

$\mathcal{L}(u) = \begin{pmatrix} s-1 & -1 \\ -1 & s \end{pmatrix} \dfrac{1}{(s)(s-1) - 1} \left(\begin{pmatrix} 1/s \\ 0 \end{pmatrix} + u_0 \right)$ $\rightarrow s^2 - s - 1$

$(s - \frac{1}{2})^2 - 5/4$

$\mathcal{L}(u) = \frac{1}{s}(s-1)\left(\frac{1}{s^2 - s - 1}\right)$ $\quad s - 1/2 = \pm\frac{\sqrt{5}}{2}$

to finish

$\left(\frac{1}{s}(-1)\left(\frac{1}{s^2 - s - 1}\right)\right) + (\ \cdot\ ) u_0$ $\quad s = \frac{1}{2} \pm \frac{\sqrt{5}}{2}$

$s = \frac{1}{2}(1 \pm \sqrt{5})$

$= \begin{pmatrix} \dfrac{s-1}{s(s^2 - s - 1)} \\ \dfrac{-1}{s(s^2 - s - 1)} \end{pmatrix}$ $\rightsquigarrow$ more

David Darais

3d) cont

$$\frac{s-1}{s\left(s-\frac{1}{2}(1+\sqrt{5})\right)\left(s-\frac{1}{2}(1-\sqrt{5})\right)}$$

$-\frac{1}{2}(1+\sqrt{5})\cdot\frac{1}{2}(1-\sqrt{5})$

$=\frac{1}{4}(1-5)=-1$

$+\frac{1}{2}(1+\sqrt{5})+\frac{1}{2}(1+\sqrt{5})$

$=-1$

$$\left(\frac{-1}{s\left(s-\frac{1}{2}(1+\sqrt{5})\right)\left(s-\frac{1}{2}(1-\sqrt{5})\right)}\right)$$

$$\left(\begin{matrix}\dfrac{\frac{1}{1}}{s}+\dfrac{\frac{\frac{1}{2}(1+\sqrt{5})-1}{\frac{1}{2}(1+\sqrt{5})(-1)}}{s-\frac{1}{2}(1+\sqrt{5})}+\dfrac{\frac{\frac{1}{2}(1-\sqrt{5})-1}{\frac{1}{2}(1-\sqrt{5})(-1)}}{s-\frac{1}{2}(1-\sqrt{5})}\\[2ex] \dfrac{\frac{-1}{-1}}{s}+\dfrac{\frac{-1}{\frac{1}{2}(1+\sqrt{5})(-1)}}{s-\frac{1}{2}(1+\sqrt{5})}+\dfrac{\frac{-1}{\frac{1}{2}(1+\sqrt{5})(-1)}}{s-\frac{1}{2}(1-\sqrt{5})}\end{matrix}\right)$$

$$u=\left(\begin{matrix}1-\left(1-\frac{1}{\frac{1}{2}(1+\sqrt{5})}\right)e^{1/2(1+\sqrt{5})t}-\left(1-\frac{1}{\frac{1}{2}(1-\sqrt{5})}\right)e^{1/2(1-\sqrt{5})t}\\[2ex] 1+\frac{1}{\frac{1}{2}(1+\sqrt{5})}e^{1/2(1+\sqrt{5})t}+\frac{1}{\frac{1}{2}(1+\sqrt{5})}e^{1/2(1-\sqrt{5})t}\end{matrix}\right)$$

atoms $e^{1/2(1+\sqrt{5})}$, $e^{1/2(1-\sqrt{5})}$

it is a ~~node~~ saddle

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4. (ch7)

(4a) [20%] Explain Laplace's Method, as applied to the differential equation $x'(t) + 2x(t) = e^t$, $x(0) = 1$.

(4b) [15%] Solve $\mathcal{L}(f(t)) = \dfrac{10}{(s^2+4)(s^2+9)}$ for $f(t)$.

(4c) [15%] Solve for $f(t)$ in the equation $\mathcal{L}(f(t)) = \dfrac{1}{s^2(s+3)}$.

(4d) [10%] Find $\mathcal{L}(f)$ given $f(t) = (-t)e^{2t}$.

(4e) [20%] Solve $x''' + x'' = 0$, $x(0) = 1$, $x'(0) = 0$, $x''(0) = 0$ by Laplace's Method.

(4f) [20%] Solve the system $x' = x + y$, $y' = x - y + 2$, $x(0) = 0$, $y(0) = 0$ by Laplace's Method.

4a. $x'(t) + 2x(t) = e^t$, $x(0) = 1$

$\mathcal{L}(x'(t) + 2x(t)) = \mathcal{L}(e^t)$ — take laplace of both sides

$\mathcal{L}(x'(t)) + 2\mathcal{L}(x(t)) = \frac{1}{s-1}$ — laplace is linear

$s\mathcal{L}(x(t)) - 1 + 2\mathcal{L}(x(t)) = \frac{1}{s-1}$ — parts rule $\mathcal{L}(x'(t)) = s\mathcal{L}(x(t)) - x(0)$

$(s+2)\mathcal{L}(x(t)) = \frac{1}{s-1} + 1$ — collect terms

$\mathcal{L}(x) = \frac{1}{(s-1)(s+2)} + \frac{1}{s+2}$ — divide

$\mathcal{L}(x) = \frac{1/3}{s-1} + \frac{2/3}{s+2}$ — partial fractions

$\mathcal{L}(x) = \mathcal{L}\left(\frac{1}{3}e^t + \frac{2}{3}e^{-2t}\right)$ — unwrap RHS

$x(t) = \frac{1}{3}e^t + \frac{2}{3}e^{-2t}$ — lerch's

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Simon Williams

4b. $\mathcal{L}(f(t)) = \dfrac{10}{(s^2+4)(s^2+9)}$. substitute $u = s^2$

$= \dfrac{10}{(u+4)(u+9)} = \dfrac{2}{u+4} - \dfrac{2}{u+9} = \dfrac{2}{s^2+4} - \dfrac{2}{s^2+9}$

$f(t) = \sin 2t - \frac{2}{3}\sin 3t$ ✓

4c. $\mathcal{L}(f(t)) = \dfrac{1}{s^2(s+3)} = \dfrac{A}{s} + \dfrac{B}{s^2} + \dfrac{C}{s+3} \Rightarrow 1 = As(s+3) + B(s+3) + Cs^2$

$C = \frac{1}{9}$ ✓

$B = \frac{1}{3}$ ✓

$A = -\frac{1}{9}$ ✓

$1 = A(1)(4) + \frac{4}{3} + \frac{1}{9}$

$1 - \frac{13}{9} = 4A$, $A = -\frac{1}{9}$

so $f(t) = -\frac{1}{9} + \frac{1}{3}t + \frac{1}{9}e^{-3t}$

4d. $\mathcal{L}(-te^{2t}) = \mathcal{L}(-t)_{s \to s-2} = -\dfrac{1}{(s-2)^2}$ ✓

4e. $x''' + x'' = 0$, $x(0) = 1$, $x'(0) = x''(0) = 0$

$\mathcal{L}(x'') = s\mathcal{L}(x')$

$\mathcal{L}(x') = s\mathcal{L}(x) - 1$

$\mathcal{L}(x''') = s\mathcal{L}(x'')$

$s(s(s\mathcal{L}(x)-1)) + s(s\mathcal{L}(x)-1) = 0$

$s^3\mathcal{L}(x) - s^2 + s^2\mathcal{L}(x) - s = 0$

$(s^3+s^2)\mathcal{L}(x) = s^2 + s$

$\mathcal{L}(x) = \dfrac{s(s+1)}{s^2(s+1)} = \dfrac{1}{s}$, $x = 1$ ✓

OVER

4f. $x' = x+y$, $x(0)=0$, $y(0)=0$
$y' = x-y+2$

$$\vec{u}' = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\vec{u} + \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

$$\mathcal{L}(\vec{u}') - \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\mathcal{L}(\vec{u}) = \mathcal{L}\begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

$$\left(sI - \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\right)\mathcal{L}(\vec{u}) = \begin{pmatrix} 0 \\ \frac{2}{s} \end{pmatrix}$$

$$\begin{pmatrix} s-1 & -1 \\ -1 & s+1 \end{pmatrix}\mathcal{L}(\vec{u}) = \begin{pmatrix} 0 \\ \frac{2}{s} \end{pmatrix}$$

$(s-1)(s+1)-1 = s^2-2$

$$\begin{pmatrix} s-1 & -1 \\ -1 & s+1 \end{pmatrix}^{-1} = \frac{1}{s^2-2}\begin{pmatrix} s+1 & 1 \\ 1 & s-1 \end{pmatrix}$$

$$\mathcal{L}(\vec{u}) = \begin{pmatrix} \frac{s+1}{s^2-2} & \frac{1}{s^2-2} \\ \frac{1}{s^2-2} & \frac{s-1}{s^2-2} \end{pmatrix}\begin{pmatrix} 0 \\ \frac{2}{s} \end{pmatrix}$$

$$\mathcal{L}(\vec{u}) = \begin{pmatrix} \frac{2}{s(s^2-2)} \\ \frac{2s-2}{s(s^2-2)} \end{pmatrix} = \begin{pmatrix} \frac{2}{s(s-\sqrt{2})(s+\sqrt{2})} \\ \frac{2(s-1)}{s(s-\sqrt{2})(s+\sqrt{2})} \end{pmatrix} = \begin{pmatrix} -\frac{1}{s} + \frac{1/2}{s-\sqrt{2}} + \frac{1/2}{s+\sqrt{2}} \\ \frac{1}{s} + \frac{\frac{(\sqrt{2}-1)}{2}}{s-\sqrt{2}} + \frac{\frac{(-\sqrt{2}-1)}{2}}{s+\sqrt{2}} \end{pmatrix}$$

$$x = -1 + \tfrac{1}{2}e^{\sqrt{2}t} + \tfrac{1}{2}e^{-\sqrt{2}t}$$

$$y = 1 + \tfrac{1}{2}(\sqrt{2}-1)e^{\sqrt{2}} + \tfrac{1}{2}(-\sqrt{2}-1)e^{-\sqrt{2}t}$$

$y(0) = 1 + \frac{1}{2}(\sqrt{2}-1) + \frac{1}{2}(-\sqrt{2}-1)$
$= 1 - \frac{1}{2} - \frac{1}{2}$
$= 0$ ✓