Name ____________________ Class Time _____

Math 2280 Extra Credit Problems
Chapter 5
S2009

Due date: Submit these problems the day after the collection of 6.4. Records are locked on that date and only corrected, never appended.

Submitted work. Please submit one stapled package per problem. Kindly label problems **Extra Credit**. Label each problem with its corresponding problem number, e.g., Xc5.1-8. You may attach this printed sheet to simplify your work.

Problem Xc5.1-12. (**Eigenpairs of a** 2×2)

Let $A = \begin{pmatrix} 9 & -10 \\ 2 & 0 \end{pmatrix}$. Find the eigenpairs of A. Then report eigenpair packages P and D such that $AP = PD$.

Problem Xc5.1-20. (**Eigenpairs of a** 3×3)

Let $A = \begin{pmatrix} 5 & -6 & 3 \\ 6 & -7 & 3 \\ 6 & -6 & 2 \end{pmatrix}$. Find the eigenpairs of A. Then report eigenpair packages P and D such that $AP = PD$.

Problem Xc5.1-32. (**Complex eigenpairs of a** 2×2)

Let $A = \begin{pmatrix} 0 & -6 \\ 24 & 0 \end{pmatrix}$. Find the eigenpairs of A. Then report eigenpair packages P and D such that $AP = PD$.

Problem Xc5.1-36. (**Eigenvalues of band matrices**)

Find the eigenvalues of the matrix A below without the aid of computers.

$$A = \begin{pmatrix} 1 & 2 & 0 & 0 & 0 & 0 \\ 2 & 1 & 2 & 0 & 0 & 0 \\ 0 & 2 & 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 & 2 & 0 \\ 0 & 0 & 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Problem Xc5.1-6. (**Eigenpair packages of a** 3×3)

Let $A = \begin{pmatrix} 2 & -2 & 1 \\ 2 & -2 & 1 \\ 2 & -2 & 1 \end{pmatrix}$. Find the eigenpairs of A. Then report eigenpair packages P and D such that $AP = PD$. Check the answer by hand, expanding both products AP and PD, finally showing equality.

Problem Xc5.1-18. (**Fourier's model for a** 3×3)

Assume Fourier's model for a certain matrix A:

$$A\left(c_1\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_3\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right) = 3c_1\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_3\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Find A explicitly from $AP = PD$. Check your answer by finding the eigenpairs of A.

Problem Xc5.1-28. (**Eigenpairs and diagonalization of a** 4×4)

Determine the eigenpairs of A below. If diagonalizable, then report eigenpair packages P and D such that $AP = PD$.

$$A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 2 & 0 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & 13 \end{pmatrix}$$

Problem Xc5.2-14. (Particular solution)

(a) Find the constants c_1, c_2 in the general solution

$$\mathbf{x}(t) = c_1 e^{3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{5t}\begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

satisfying the initial conditions $x_1(0) = 4$, $x_2(0) = -1$.

(b) Find the matrix A in the equation $\mathbf{x}' = A\mathbf{x}$. Use the formula $AP = PD$ and Fourier's model for A, which is given implicitly in (a) above, and explicitly as

$$A(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2$$

where c_1, c_2 are arbitrary constants and $(\lambda_1, \mathbf{v}_1)$, $(\lambda_2, \mathbf{v}_2)$ are the eigenpairs of the 2×2 matrix A.

Problem Xc5.2-8. (Eigenanalysis method 2×2)

(a) Find λ_1, λ_2, $\mathbf{v}_1$, $\mathbf{v}_2$ in Fourier's model $A\,(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2$ for

$$A = \begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix}.$$

(b) Display the general solution of $\mathbf{x}' = A\mathbf{x}$.

Problem Xc5.2-20. (Eigenanalysis method 3×3)

(a) Find λ_1, λ_2, λ_3, $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ in Fourier's model $A\,(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3) = c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2 + c_3\lambda_3\mathbf{v}_3$ for

$$A = \begin{pmatrix} 2 & 1 & -1 \\ -4 & -3 & -1 \\ 4 & 4 & 2 \end{pmatrix}.$$

(b) Display the general solution of $\mathbf{x}' = A\mathbf{x}$.

Problem Xc5.2-30. (Brine Tanks)

Consider two brine tanks satisfying the equations

$$x_1'(t) = -k_1x_1 + k_2x_2, \quad x_2' = k_1x_1 - k_2x_2.$$

Assume $r = 10$ gallons per minute, $k_1 = r/V_1$, $k_2 = r/V_2$, $x_1(0) = 30$ and $x_2(0) = 0$. Let the tanks have volumes $V_1 = 50$ and $V_2 = 25$ gallons. Solve for $x_1(t)$ and $x_2(t)$.

Problem Xc5.2-40. (Eigenanalysis method 4×4)

Display (a) Fourier's model and (b) the general solution of $\mathbf{x}' = A\mathbf{x}$ for the 4×4 matrix

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ -21 & -5 & -27 & -9 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & -16 & -4 \end{pmatrix}.$$

Problem Xc5.4-4. (Fundamental Matrix)

Consider the 2×2 vector-matrix differential equation

$$\mathbf{u}' = A\mathbf{u}, \quad A = \begin{pmatrix} 2 & -5 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

Complete all parts below.

(a) *Cayley-Hamilton method.* Compute the characteristic equation $\det(A - \lambda I) = 0$. Find two atoms from the roots of this equation. Then $x(t)$ is a linear combination of these atoms. The first equation $x' = 2x - 5y$ can be solved for y to find the second answer. Construct a fundamental matrix Φ from these scalar answers.

(b) *Eigenanalysis method.* Find the eigenpairs $(\lambda_1, \mathbf{v}_1), (\lambda_2, \mathbf{v}_2)$ of A. Let Φ have columns $e^{\lambda_1 t}\mathbf{v}_1,\ e^{\lambda_2 t}\mathbf{v}_2$. Explain why Φ is a fundamental matrix.

(c) *Putzer's formula.* Find e^{At} from the formula

$$e^{At} = e^{\lambda_1 t}T + \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}(A - \lambda_1 I).$$

If the eigenvalues are complex, then e^{At} is the real part of the right side. If $\lambda_1 = \lambda_2$, then E^{At} is the limit of the right side as $\lambda_2 \to \lambda_1$ (use L'Hopital's rule).

(d) Report $e^{At} = \Phi(t)\Phi(0)^{-1}$, using the answer for Φ from part (a) or (b). Check your answer against the one in part (c).

Problem Xc5.5-12. (Putzer's Method)

The exponential matrix e^{At} can be found in the 2×2 case from Putzer's formula

$$e^{At} = e^{\lambda_1 t}I + \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}(A - \lambda_1 I).$$

If the roots λ_1, λ_2 of $\det(A - \lambda I) = 0$ are equal, then compute the Newton quotient factor by L'Hopital's rule, limiting $\lambda_2 \to \lambda_1$ [λ_1, t fixed]. If the roots are complex, then take the real part of the right side of the equation.

Compute e^{At} from Putzer's formula for the following cases.

(a) $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. Answer $e^{At} = \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix}$.

(b) $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

(c) $A = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}$.

(d) $A = \begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix}$.

Problem Xc5.4-38. (Laplace's Resolvent Method)

The exponential matrix e^{At} can be found from the Laplace resolvent formula for the problem $\Phi' = A\Phi$, $\Phi(0) = I$:

$$\mathcal{L}(\Phi(t)) = (sI - A)^{-1}\Phi(0) = (sI - A)^{-1}.$$

For example, $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ gives $\mathcal{L}(e^{At}) = \begin{pmatrix} s-1 & 0 \\ 0 & s-2 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{s-1} & 0 \\ 0 & \frac{1}{s-2} \end{pmatrix} = \begin{pmatrix} \mathcal{L}(e^t) & 0 \\ 0 & \mathcal{L}(e^{2t}) \end{pmatrix}$, which implies $e^{At} = \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix}$.

Compute $\Phi(t) = e^{At}$ using the resolvent formula for the following cases.

(a) $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

(b) $A = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}$.

(c) $A = \begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix}$.

Problem Xc5.6-4. (Variation of Parameters)

Use the variation of parameters formula $\mathbf{u}_p(t) = e^{At}\int e^{-At}\mathbf{f}(t)dt$ to find a particular solution of the given system. Please use `maple` to do the indicated integration, following the example below.

(a) $\mathbf{u}' = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \mathbf{u} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

(b) $\mathbf{u}' = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} \mathbf{u} + \begin{pmatrix} e^t \\ 1 \end{pmatrix}$.

Example: Solve for $\mathbf{u}_p(t)$: $\mathbf{u}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{u} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

```
with(LinearAlgebra):
A:=Matrix([[0,1],[1,0]]);
f:=t->Vector([1,0]);
expAt:=t->MatrixExponential(A,t);
integral:=Map(g->int(g,t),expAt(-t).f(t));
up:=simplify(expAt(t).integral);
```

Problem Xc5.6-19. **(Initial Value Problem)**

Solve the given initial value problem using a computer algebra system. Follow the example given below.

(a) $\mathbf{u}' = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \mathbf{u} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\mathbf{u}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

(b) $\mathbf{u}' = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} \mathbf{u} + \begin{pmatrix} e^t \\ 1 \end{pmatrix}$, $\mathbf{u}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Example: Solve for $\mathbf{u}(t)$: $\mathbf{u}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{u} + \begin{pmatrix} -1 \\ 0 \end{pmatrix}$, $\mathbf{u}(0) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$. The answer is $\mathbf{u} = \begin{pmatrix} -e^{-t} \\ e^{-t} - 1 \end{pmatrix}$.

```
with(LinearAlgebra):
A:=Matrix([[0,1],[1,0]]);
f:=t->Vector([1,0]);
expAt:=t->MatrixExponential(A,t);
integral:=Map(g->int(g,t=0..t),expAt(-t).f(t));
up:=unapply(expAt(t).integral,t):
u0:=Vector([-1,0]);
uh:=t->expAt(t).(u0-up(0));
u:=simplify(uh(t)+up(t));
```

End of extra credit problems chapter 5.