Math 2250 Maple Project 7: Laplace Applications
S2009

Due date: See the internet due dates. Maple lab 7 has five problems L7.1, L7.2, L7.3, L7.4, L7.5.


Problem L7.1. (Periodic Wave Plots)

In the table are examples of standard periodic waves.

(a) Plot them all. Please choose an appropriate graph window for each.

(b) Piecewise expressions $h$ are given in the table on the base interval $[0, T]$. The $T$-periodic extension of $f$ off the base interval is always $h(g(t))$ where $g(t) = t - T \lfloor t/T \rfloor$. Observe that $g(t)$ equals $T f_2(t/T)$; see the table. Justify every maple expression in the table. In the first example, define $h_1 := t \rightarrow \text{piecewise}(t<1,1,t<2,-1,0)$ and then plot $h_1(T*f_2(t/T))-f_1(t)$ over 3 periods $[T = 2$ for the square wave]. It should plot as the zero function.

Useful plot options are ytickmarks=3, color=red, labels=[t,'f(t)'], title="square wave", numpoints=100, thickness=2. Combine options like this: $\text{opts} := \text{discont=true}, \text{thickness=2}$ and use them as $\text{plot}(f(t),t=a..b,\text{opts};$.

<table>
<thead>
<tr>
<th>Maple Expression</th>
<th>Name</th>
<th>T</th>
<th>Piecewise Definition on $[0, T]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_1 := t \rightarrow (-1)^\lfloor t \rfloor$;</td>
<td>square wave</td>
<td>2</td>
<td>$h_1(t) = \begin{cases} 1 &amp; 0 \leq t &lt; 1, \ -1 &amp; 1 \leq t &lt; 2 \end{cases}$</td>
</tr>
<tr>
<td>$f_2 := t \rightarrow t-\lfloor t \rfloor$;</td>
<td>triangular wave</td>
<td>1</td>
<td>$h_2(t) = \begin{cases} t &amp; 0 \leq t &lt; 1, \ 0 &amp; t = 1 \end{cases}$</td>
</tr>
<tr>
<td>$f_3 := t \rightarrow 1/2+(f_2(t)-1/2)*f_1(t)$;</td>
<td>sawtooth wave</td>
<td>2</td>
<td>$h_3(t) = \begin{cases} t &amp; 0 \leq t &lt; 1, \ 2-t &amp; 1 \leq t &lt; 2 \end{cases}$</td>
</tr>
<tr>
<td>$f_4 := t \rightarrow</td>
<td>\sin(t)</td>
<td>$;</td>
<td>rectified sine</td>
</tr>
<tr>
<td>$f_5 := t \rightarrow \sin(t)+abs(sin(t))$;</td>
<td>half-wave rectified sine</td>
<td>$2\pi$</td>
<td>$h_5(t) = \begin{cases} \sin(t) &amp; 0 \leq t &lt; \pi, \ 0 &amp; \pi \leq t &lt; 2\pi \end{cases}$</td>
</tr>
<tr>
<td>$p := t \rightarrow (2-t)*t$;</td>
<td>parabolic wave</td>
<td>4</td>
<td>$h_6(t) = \begin{cases} p(t) &amp; 0 \leq t &lt; 2, \ -p(t-2) &amp; 2 \leq t &lt; 4 \end{cases}$</td>
</tr>
<tr>
<td>$q := t \rightarrow \text{piecewise}(t&lt;Pi,\sin(t),t&lt;2*Pi,-1);$</td>
<td>piecewise sine pulse</td>
<td>$2\pi$</td>
<td>$h_7(t) = \begin{cases} \sin(t) &amp; 0 \leq t &lt; \pi, \ -1 &amp; \pi \leq t &lt; 2\pi \end{cases}$</td>
</tr>
</tbody>
</table>

Problem L7.2. (Hammer Hit Oscillation)

An attached mass in an undamped spring-mass system is released from rest 1 meter below the equilibrium position. After 3 seconds of oscillation, the mass is struck by a hammer with force of 5 Newtons in a downward direction.

(a) Assume the model

$$\frac{d^2 x}{dt^2} + 9x = 5\delta(t-3); x(0) = 1, \frac{dx}{dt}(0) = 0,$$

where $x(t)$ denotes the displacement from equilibrium at time $t$ and $\delta(t-3)$ denotes the Dirac delta function. Determine, using the dsolve example below, a piecewise-defined formula for $x(t)$. Plot $x(t)$ for $0 \leq t \leq 7$.

(b) Solve the following hammer-hit models DE1 to DE4, given as maple expressions, using the dsolve example for DE, IC as a template for the solution.

(c) Express the symbolic answer for each of DE1 to DE4 as a piecewise-defined function. Interpret each answer physically.

DE:=diff(x(t),t,t)+9*x(t)=3*Dirac(t-3); IC:=x(0)=1,D(x)(0)=0;
dsolve({DE,IC},x(t),method=laplace);
# x(t) = cos(3*t)+Heaviside(t-3)*sin(-9+3*t)
convert(%,piecewise);combine(%,trig);
# x(t) = cos(3*t) for t < 3, cos(3*t)+sin(-9+3*t) for t>3, undef t=3.

DE1:=diff(x(t),t,t)+9*x(t)=5*Dirac(t-3); IC1:=x(0)=-1,D(x)(0)=1;
DE2:=diff(x(t),t,t)+9*x(t)=6*Dirac(t-3); IC2:=x(0)=1,D(x)(0)=-1;
DE3:=diff(x(t),t,t)+9*x(t)=8*Dirac(t-3); IC3:=x(0)=0,D(x)(0)=-1;
DE4:=diff(x(t),t,t)+9*x(t)=9*Dirac(t-3); IC4:=x(0)=1,D(x)(0)=0;

Problem L7.3. (Maple Solution of Initial Value Problems)
(a) Solve the IVP $y'' - y' - 2y = 5 \sin x$, $y(0) = 1$, $y'(0) = -1$. Please use the inttrans package. Show the steps in Laplace’s method, entirely in maple, with explicit use of maple functions laplace(f,t,s) and intlaplace(F,s,t).
(b) Solve the pulse-input IVP
\[
3y'' + 3y' + 2y = \begin{cases} 
0 & \text{for } t < 0, \\
3 & \text{for } 0 \leq t < 4, \\
0 & \text{for } t \geq 4,
\end{cases}
\]
with initial data $y(0) = 0$, $y'(0) = 0$. Use any maple method. Express your answer as a piecewise-defined function.
(c) Solve the IVP $y'' + y = 1 + \delta(t-2\pi)$, $y(0) = 1$, $y'(0) = 0$. Use maple dsolve. Express the answer as a piecewise-defined function.

Problem L7.4. (Expressions for Periodic Waves)
Let $h$ be the $T$-periodic extension of $f(x) = 2/10 + (7/10) \sin x + (1/10) \cos 5x$, defined on $[0, T]$, for $T = 2$.
(a) Plot $h(t)$ on the interval $[-10, 10]$. Use the composition formula $h(t) = f(g(t))$, where $g(t) = t - T \lfloor t/T \rfloor$.
(b) Compute the Laplace of $h(t)$ directly from the periodic function theorem, using the sample maple code
\[
\text{int}(f(g(t)) \exp(-s*t), t=0..T)/(1-\exp(-s*T));
\]
Replacing $f(x)$ by $(1/10) \cos(5x)$ should give the answer below. The answer for $2/10 + (7/10) \sin x + (1/10) \cos 5x$ has many more terms.
\[
\frac{1}{10} \frac{se^{2s} - s \cos(10) + 5 \sin(10)}{s^2 + 25} \frac{(-1 + e^{2s})}{(-1 + e^{2s})}
\]
(c) Maple finds the laplace of $g(t) = t - T \lfloor t/T \rfloor$, but not the laplace of $h(t) = f(g(t))$. Express $h(t)$ as a series of pulses to get help from maple. The laplace of some terms in the series $h(t)$ can be computed, but not all. The constant term and sine term are a success but the cosine term causes maple to churn. This example shows that the periodic function theorem is a basic tool in Laplace theory. Here’s the success story for this example:

\[
pulse:=(t,a,b)->\text{Heaviside}(t-a)-\text{Heaviside}(t-b);
f := x \to 2/10+7/10*\sin(x): h := t->\text{sum}(f(t-n*T)*\pulse(t,n*T,n*T+T),n=0..\infty);
\text{inttrans}[\text{laplace}](h(t),t,s);
\text{eval}(\%) \text{ assuming } n::\text{positive};
\]

Here’s what does not work. See if you can change the code and make it work, giving the answer in (b) above. Beware of testing the code below: it uses about 800mb memory and finishes with no answer.

\[
pulse:=(t,a,b)->\text{Heaviside}(t-a)-\text{Heaviside}(t-b);
f := x \to (1/10)*\cos(5*x):
h := t->\text{sum}(f(t-n*T)*\pulse(t,n*T,n*T+T),n=0..\infty);
\text{inttrans}[\text{laplace}](h(t),t,s);
\text{eval}(\%) \text{ assuming } n::\text{positive};
\]
Problem L7.5. (Resolvent Method)

The Laplace resolvent formula for the problem \( u' = Au, \ u(0) = u_0 \) is

\[
\mathcal{L}(u(t)) = (sI - A)^{-1} u_0.
\]

For example, \( A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \) gives

\[
\mathcal{L}(u(t)) = \begin{pmatrix} s - 1 & 0 \\ 0 & s - 2 \end{pmatrix}^{-1} u_0 = \begin{pmatrix} \frac{1}{s-1} & 0 \\ 0 & \frac{1}{s-2} \end{pmatrix} u_0 = \begin{pmatrix} \mathcal{L}(e^t) & 0 \\ 0 & \mathcal{L}(e^{2t}) \end{pmatrix} u_0,
\]

which implies \( u(t) = \begin{pmatrix} e^t \\ 0 \\ e^{2t} \end{pmatrix} u_0. \)

The answers for the components of \( u \) are \( \alpha e^t, \beta e^{2t} \), according to the following maple code:

```maple
with(LinearAlgebra):with(inttrans):
A:=Matrix([[1,0],[0,2]]);
u0:=Vector([alpha,beta]);
B:=(s*IdentityMatrix(2)-A)^(-1).u0;
u:=Map(invlaplace,B,s,t);
```

Compute the solution \( u(t) \) using the resolvent formula for the following cases.

(a) \( A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \ u(0) = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \)

(b) \( A = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}, \ u(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \)

(c) \( A = \begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix}, \ u(0) = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \)