Differential Equations and Linear Algebra

2250-1 [7:30 class] at 8:00am on 5 May 2009

Instructions. The time allowed is 120 minutes. The examination consists of eight problems, one for each of chapters 3, 4, 5, 6, 7, 8, 9, 10, each problem with multiple parts. A chapter represents 15 minutes on the final exam.

Each problem on the final exam represents several textbook problems numbered (a), (b), (c), $\cdots$. Choose the problems to be graded by check-mark $\boxed{\text{X}}$. The credits should add to 100. Each chapter (3 to 10) adds at most 100 towards the maximum final exam score of 800. There may be replacement problems at reduced credit, in case the problem (a), (b), (c), $\cdots$ cannot be solved. The number of graded problems is fixed. For instance, if ch3(a) and ch3(a_1) are solved, then ch3(a) is ignored and its replacement ch3(a_1) will be graded instead. The final exam grade is reported as a percentage 0 to 100, as follows:

$$\text{Final Exam Grade} = \frac{\text{Sum of scores on eight chapters}}{8}.$$

- Calculators, books, notes and computers are not allowed.
- Details count. Less than full credit is earned for an answer only, when details were expected. Generally, answers count only 25% towards the problem credit.
- Completely blank pages count 40% or less, at the whim of the grader.
- Answer checks are not expected and they are not required. First drafts are expected, not complete presentations.
- Please prepare **exactly eight** separately stapled packages of problems, one package per chapter. These packages will be submitted as four grading stacks, no extra staples, for grading by

  Tam [ch3,ch4], **Gakkhar** [ch5,ch6], **Nguyen** [ch7,ch8], **Gustafson** [ch9,ch10].

  Each grader will add one staple to bind the chapter packages. The graded exams will be in a box outside 113 JWB; you will pick up 4 stapled packages.

Final Grade. The final exam counts as two midterm exams. For example, if exam scores earned were 90, 91, 92 and the final exam score is 89, then the exam average for the course is

$$\text{Exam Average} = \frac{90 + 91 + 92 + 89 + 89}{5} = 90.2.$$

Dailies count 30% of the final grade. The course average is computed from the formula

$$\text{Course Average} = \frac{70}{100}(\text{Exam Average}) + \frac{30}{100}(\text{Dailies Average}).$$

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Ch3. (Linear Systems and Matrices) Check the boxes ☐ on the **three problems** to be graded, which is 100%. Label worked problems accordingly.

☐ [40%] Ch3(a): This problem uses the identity $A\,\mathbf{adj}(A) = \mathbf{adj}(A)A = |A|I$, where $|A|$ is the determinant of matrix A. Symbol $\mathbf{adj}(A)$ is the adjugate or adjoint of A. The identity is used to derive the adjugate inverse identity $A^{-1} = \mathbf{adj}(A)/|A|$, a topic in Section 3.6 of Edwards-Penney.

Let B be the matrix given below, where $\boxed{?}$ means the value of the entry does not affect the answer to this problem. The second matrix is $C = \mathbf{adj}(B)$. Report the value of the determinant of matrix $BC^{-1}B$.

$$B = \begin{pmatrix} 1 & ? & -1 & 0 \\ 1 & ? & 0 & 0 \\ ? & 0 & 2 & ? \\ ? & 0 & 0 & ? \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 4 & 0 & 0 \\ -4 & 4 & -2 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

☐ [40%] Ch3(b): Determine which values of k correspond to (1) a unique solution, (2) infinitely many solutions and (3) no solution, for the system $A\mathbf{x} = \mathbf{b}$ given by

$$A = \begin{pmatrix} 1 & 3 & -k \\ 0 & k-2 & k-4 \\ 1 & 3 & -4 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 2k-6 \\ k-3 \end{pmatrix}.$$

☐ [20%] Ch3(c): Let matrix C and vector $\mathbf{b}$ be defined by the equations

$$C = \begin{pmatrix} -1 & 3 & -1 \\ 0 & -1 & 4 \\ 1 & 3 & -3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}.$$

Let I denote the 3×3 identity matrix. Find the value of x_3 by Cramer's Rule in the system $(I+C)\mathbf{x} = \mathbf{b}$.

If you finished Ch3(a), Ch3(b) and Ch3(c), then 100% has been marked – go on to Ch4. Otherwise, replace Ch3(a) with Ch3(a_1) and/or Ch3(b) by Ch3(b_1). **A maximum of three problems will be graded.** There is reduced credit for each replacement.

☐ [30%] Ch3(a_1): Find the adjugate of A and the inverse of $A = \begin{pmatrix} 0 & 2 & -1 \\ 0 & 0 & 4 \\ 1 & 3 & -2 \end{pmatrix}$.

☐ [30%] Ch3(b_1):

Part I [10%]: State three row operations which are used to solve a linear system $A\mathbf{x} = \mathbf{b}$.

Part II [10%]: Give an example of two 2×2 matrices A and B such that AB is lower triangular.

Part III [10%]: Give an example of a 3×3 matrix A such that the system $A\mathbf{x} = \mathbf{0}$ has solutions $\mathbf{x} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$ and $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$.

Staple this page to the top of all Ch3 work. Submit one stapled package per chapter.

Ch3 (a) row(B,3) col(C,3) = |B| = 4, $|BC| = |4I| = 4^4$, $|C| = 4^3$,

$$|BC^{-1}B| = |B|^2/|C| = \frac{4^2}{4^3} = \boxed{\frac{1}{4}}$$

Ch3 (b) combo(1,3,−1) ⇒ $\left(\begin{array}{ccc|c} 1 & 3 & -k & 1 \\ 0 & k-2 & k-4 & 2k-6 \\ 0 & 0 & k-4 & k-3-1 \end{array}\right)$ a triangular form

$k=4$ ∞-many sols (row of zeros, 3x3 problem)
$k=2$ ∞-many sols (one free variable)
$(k-2)(k-4) \neq 0$ unique sol (matrix is invertible)
No signal equation ⟺ No sol Never happens

Ch3 (c) $I+C = \begin{pmatrix} 0 & 3 & -1 \\ 0 & 0 & 4 \\ 1 & 3 & -2 \end{pmatrix}$ $\Delta = -4\begin{vmatrix} 0 & 3 \\ 1 & 3 \end{vmatrix} = 12$ $\Delta_3 = \begin{vmatrix} 0 & 3 & 0 \\ 0 & 0 & 2 \\ 1 & 3 & 1 \end{vmatrix} = -3\begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix} = 6$

$$x_3 = \frac{\Delta_3}{\Delta} = \frac{6}{12} = \boxed{\frac{1}{2}}$$

Ch3 (a_1) $\text{adj}(A) = \begin{pmatrix} -12 & 1 & 8 \\ 4 & 1 & 0 \\ 0 & 2 & 0 \end{pmatrix}$, $A^{-1} = \frac{1}{4}\,\text{adj}(A)$

Ch3 (b_1)

I. Combo, swap, mult

II. $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$

III. $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

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Ch4. (Vector Spaces) Check the boxes ☐ on the **four problems** to be graded, which is 100%. Label worked problems accordingly.

☐ [30%] Ch4(a): Define S to be the set of all vectors $\mathbf{x}$ in $\mathcal{R}^4$ such that $x_1 + x_3 + 2x_4 = x_2$ and $x_2 + x_3 = 0$. Prove that S is a subspace of $\mathcal{R}^4$.

☐ [10%] Ch4(b): Independence of 3 fixed vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ can be decided by counting the pivot columns of their augmented matrix. State a **different** test which can decide upon independence of three vectors in $\mathcal{R}^4$.

☐ [30%] Ch4(c): Consider the four vectors

$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}, \quad \mathbf{w}_1 = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}, \quad \mathbf{w}_2 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}.$$

The subspaces $S_1 = \mathbf{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ and $S_2 = \mathbf{span}\{\mathbf{w}_1, \mathbf{w}_2\}$ each have dimension 2 and share a common vector $\mathbf{v}_2 = \mathbf{w}_1$. Explain why S_1 is not equal to S_2.

☐ [30%] Ch4(d): The 5×6 matrix A below has some independent columns. Report the independent columns of A, according to the Pivot Theorem.

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & -2 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 6 & 0 & 6 & 0 & 0 \\ 0 & 2 & 0 & 2 & 0 & 0 \end{pmatrix}$$

If you marked Ch4(a) through Ch4(d), then 100% has been marked – go on to Ch5. **Otherwise**, mark replacement problems from the possibilities Ch4(a) → Ch4(a_1), Ch4(c) → Ch4(c_1). **A maximum of four problems will be graded**. Replacements have reduced credit.

☐ [25%] Ch4(a_1): State a determinant test whose conclusion is the independence of three functions $f_1(x)$, $f_2(x)$, $f_3(x)$.

☐ [25%] Ch4(c_1): Apply an independence test to the vectors below. Report **independent** or **dependent** .

$$\mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 2 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 4 \\ -1 \\ -1 \\ 0 \end{pmatrix}.$$

Staple this page to the top of all Ch4 work. Submit one stapled package per chapter.

ch4 (a) Let $A = \begin{pmatrix} 1 & -1 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Then $A\vec{x} = \vec{0}$ defines S.

Apply the kernel Thm. Then S is a subspace.

Ch 4 (b) Independent $V_1, V_2, V_3 \iff \text{rank}(\text{aug}(v_1, v_2, v_3)) = 3$

Ch 4 (c) $\text{rank}\begin{pmatrix} 2 & 1 & 2 \\ 1 & 2 & -1 \\ -3 & -3 & 0 \end{pmatrix} = 3 \Rightarrow$ indep. Cols. $\Rightarrow \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$ not in S

Ch 4 (d) Cols 2,4 are indep. The others are linear combinations of those 2 cols.

Ch 4 (a_1) Wronskian Test: $\begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix} \neq 0$ at some $x \Rightarrow f_1, f_2, f_3$ indep.

Sample Test: choose x_1, x_2, x_3.

$\begin{vmatrix} f_1(x_1) & f_2(x_1) & f_3(x_1) \\ f_1(x_2) & f_2(x_2) & f_3(x_2) \\ f_1(x_3) & f_2(x_3) & f_3(x_3) \end{vmatrix} \neq 0 \Rightarrow f_1, f_2, f_3$ indep.

A hybrid of these 2 also is a valid test

Ch4 (c_1) $\text{rank}\begin{pmatrix} -1 & 3 & 4 \\ 1 & 0 & -1 \\ 2 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} = 2 \Rightarrow$ dependent cols

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Ch5. (Linear Equations of Higher Order) Solve five problems. Problem Ch5(e) is required.

☐ [20%] Ch5(a): Report the general solution $y(x)$ of the differential equation

$$7\frac{d^2y}{dx^2} + 36\frac{dy}{dx} + 5y = 0.$$

☐ [20%] Ch5(b): Given a damped spring-mass system $mx''(t) + cx'(t) + kx(t) = 0$ with $m = 3$, $c = 10$ and $k > 0$ a symbol, calculate all values of k such that the solution $x(t)$ is **over-damped**. Please, **do not solve** the differential equation!

☐ [20%] Ch5(c): Find the characteristic equation of a higher order linear homogeneous differential equation with constant coefficients such that $y = x + e^{-x} + \cos 2x$ is a solution.

☐ [20%] Ch5(d): Determine the general solution $y(x)$ of the homogeneous constant-coefficient differential equation, given it has characteristic equation

$$r(r^2 - r)^2(r^2 + 11) = 0.$$

☐ [20%] Ch5(e): **No replacement. This problem is required.**
Determine the **shortest** trial solution for y_p according to the method of undetermined coefficients. **Do not evaluate** the undetermined coefficients!

$$\frac{d^4y}{dx^4} + \frac{d^3y}{dx^3} = x + 3\cos x + 4e^{-x}$$

If you marked Ch5(a) through Ch5(e), then 100% has been marked - go on to Ch5. **Otherwise**, unmark one problem from Ch5(a) to Ch5(d) and complete the replacement problem Ch5(f). **A maximum of five problems will be graded.**

☐ [20%] Ch5(f): **Unmark one of (a) to (d) above. Only five will be graded. This problem cannot replace Ch5(e).**
A particular solution of the differential equation $x'' + 4x' + 15x = 50\cos(5t)$ is

$$x(t) = -\cos 5t + 15e^{-2t}\sin\sqrt{11}t + 2\sin 5t.$$

Identify the **transient** solution x_{tr} and the **steady-state** periodic solution $x_{\text{SS}}(t)$.

Staple this page to the top of all Ch5 work. Submit one stapled package per chapter.

Ch 5 (a) $7r^2+36r+5=0$ $\qquad y = c_1 e^{-x/7} + c_2 e^{-5x}$

$(7r+1)(r+5)=0$

Ch 5 (b) $c^2-4km = 100-12k > 0 \Rightarrow$ over-damped

Ch 5 (c) atoms $1, x, e^{-x}, \cos 2x, \sin 2x$ required

roots $0, 0, -1, \pm 2i$

$r^2(r+1)(r^2+4)=0$ will work

Ch 5 (d) $r^3(r-1)^2(r^2+11)=0$

$0,0,0,1,1, \pm\sqrt{11}\, i$

atoms $= 1, x, x^2, e^x, xe^x, \cos(\sqrt{11}\,x), \sin(\sqrt{11}\,x)$

$y(x)=$ linear combination of these 7 atoms

Ch 5 (e) $r^4+r^3=0$

$r^3(r+1)=0$ roots $0,0,0,-1$

$f(x) = x + 3\cos x + 4e^{-x}$

atoms for $f = 1, x, \cos x, \sin x, e^{-x}$

Trial sol $y =$ linear combination of these atoms, initially

Shortest trial sol $y =$ linear combination of 5 atoms below

group 1: $\boxed{1}, \boxed{x}, \boxed{x^2}, x^3, x^4$

remove boxed atoms

group 2: $\cos x, \sin x$

group 3: $\boxed{e^{-x}}, xe^{-x}$

$\left.\right\rangle$ $x^3, x^4, \cos x, \sin x, xe^{-x}$ — These atoms are used to form the shortest trial sol

Ch 5 (f) transient $= 15e^{-2t}\sin(\sqrt{11}\,t)$

Steady-state $= 2\sin 5t - \cos 5t$

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Ch6. (Eigenvalues and Eigenvectors) Check the boxes ☐ on the **three problems** to be graded, which is 100%. Label worked problems accordingly.

☐ [50%] Ch6(a): **This problem is required. No replacement.**

Find the eigenvalues of the matrix $A = \begin{pmatrix} 0 & 5 & -5 & 0 & 0 \\ -5 & 0 & -12 & 3 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 5 & 1 & 4 \end{pmatrix}$.

To save time, **do not** find eigenvectors!

☐ [25%] Ch6(b): Let A be a 2×2 matrix satisfying for all real numbers c_1, c_2 the identity

$$A\left(c_1\begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2\begin{pmatrix} -1 \\ 3 \end{pmatrix}\right) = 3c_1\begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

Find a diagonal matrix D and an invertible matrix P such that $AP = PD$.

☐ [25%] Ch6(c): Find a 2×2 matrix A with eigenpairs

$$\left(5, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right), \quad \left(-4, \begin{pmatrix} 1 \\ 3 \end{pmatrix}\right).$$

If you finished Ch6(a), Ch6(b) and Ch6(c), then 100% has been marked – go on to Ch4. **Otherwise**, unmark one or two of Ch6(b), Ch6(c) and go on to complete Ch6(d) and/or Ch6(e). **Only three problems will be graded.**

☐ [25%] Ch6(d): Assume two 3×3 matrices A, B are related by $AP = PB$ where P is invertible. Let A have eigenvalues 1, 4, 6. Find the eigenvalues of $3B + 2I$, where I is the identity matrix.

☐ [25%] Ch6(e): Let A be a 3×3 matrix with eigenpairs

$$(4, \mathbf{v}_1), \quad (3, \mathbf{v}_2), \quad (1, \mathbf{v}_3).$$

Let P denote the augmented matrix of the eigenvectors $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_1$, in exactly that order. Display the answer for $P^{-1}AP$. Justify the answer with a sentence.

Staple this page to the top of all Ch6 work. Submit one stapled package per chapter.

Ch6 (a) Char. poly. $= |A - \lambda I| = (4-\lambda)\big((1-\lambda)(3-\lambda)+1\big)(\lambda^2+25)$

$\lambda = 4, 2, 2, 5i, -5i$

Ch6 (b) Take $c_1 = 1$, $c_2 = 0$. Then $A\begin{pmatrix}1\\-2\end{pmatrix} = 3\begin{pmatrix}-1\\2\end{pmatrix} = (-3)\begin{pmatrix}1\\-2\end{pmatrix} \Longrightarrow$

$\left(-3, \begin{pmatrix}1\\-2\end{pmatrix}\right)$ = eigenpair. Take $c_1 = 0$, $c_2 = 1$. Then $A\begin{pmatrix}-1\\3\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$

$\Rightarrow \left(0, \begin{pmatrix}-1\\3\end{pmatrix}\right)$ = eigenpair. Let $D = \begin{pmatrix}-3 & 0\\0 & 0\end{pmatrix}$, $P = \begin{pmatrix}1 & -1\\-2 & 3\end{pmatrix}$.

Ch6 (c) $A = \begin{pmatrix}1 & 1\\2 & 3\end{pmatrix}\begin{pmatrix}5 & 0\\0 & -4\end{pmatrix}\begin{pmatrix}1 & 1\\2 & 3\end{pmatrix}^{-1}$

Ch6 (d) $|3B + 2I - \lambda I| = |3(B + \frac{2}{3}I - \frac{\lambda}{3}I)| = |3(B - (\frac{\lambda}{3} - \frac{2}{3})I)|$

$\Rightarrow \frac{\lambda}{3} - \frac{2}{3}$ = Eigenvalue of B = Eigenvalue of $A = 1, 4, 6$

$\lambda = 3\left(1 + \frac{2}{3}, 4 + \frac{2}{3}, 6 + \frac{2}{3}\right) = 5, 14, 20$

Ch (e) $AP = PD \Rightarrow D = P^{-1}AP \Rightarrow D = \begin{pmatrix}3 & 0 & 0\\0 & 1 & 0\\0 & 0 & 4\end{pmatrix}$ because of the order in P of v_2, v_3, v_1

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Ch7. (Linear Systems of Differential Equations) Check the boxes ☐ on the **three problems** to be graded, which is 100%. Label worked problems accordingly.

☐ [40%] Ch7(a): **No replacement. This problem is required.**
Solve for the general solution $x(t)$, $y(t)$ in the system below. Use any method that applies, from the lectures or any chapter of the textbook.

$$\begin{aligned} \frac{dx}{dt} &= x + y, \\ \frac{dy}{dt} &= 6x + 2y. \end{aligned}$$

☐ [40%] Ch7(b): **No replacement. This problem is required.**
Apply the eigenanalysis method to solve the differential system $\mathbf{u}' = A\mathbf{u}$, given

$$A = \begin{pmatrix} 12 & -5 & 5 \\ 0 & 2 & 0 \\ -10 & 5 & -3 \end{pmatrix}$$

The eigenvalues of A are 2, 2, 7. The term *eigenanalysis* refers to the process of finding eigenvalues and eigenvectors of the matrix A. After finding the eigenpairs, report the general solution $\mathbf{u}(t)$.

☐ [20%] Ch7(c): Assume A is a 2×2 matrix and the general solution of $\mathbf{u}' = A\mathbf{u}$ is given by

$$\mathbf{u}(t) = c_1 e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

Find the matrix A.

If you marked Ch7(a), Ch7(b) and Ch7(c), then 100% has been marked – go on to Ch8. **Otherwise**, you may replace problem Ch7(c) by Ch7(c_1). A maximum of three problems will be graded. The replacement has reduced credit.

☐ [15%] Ch7(c_1): A 2×2 real matrix A has eigenvalues 1 and 2. Display the form of the general solution of the differential equation $\mathbf{u}' = A\mathbf{u}$.

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Ch 7 (a) $|A - rI| = \begin{vmatrix} 1-r & 1 \\ 6 & 2-r \end{vmatrix} = r^2 - 3r - 4 = (r-4)(r+1) \Rightarrow$ roots $= -1, 4$

$\boxed{x(t) = c_1 e^{-t} + c_2 e^{4t}}$, by Cayley-Hamilton method, which implies components are l.c. of atoms e^{-t}, e^{4t}.

$y(t) = x' - x$ from eq #1

$\boxed{y(t) = -2c_1 e^{-t} + 3c_2 e^{4t}}$

Other methods: Laplace, Cayley-Hamilton with $\vec{c}_1, \vec{c}_2$, e^{At}, Eigenanalysis.

Ch 7 (b) Eigenpairs $= \left(2, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right), \left(2, \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}\right), \left(7, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}\right)$

$$\vec{u}(t) = c_1 e^{2t} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + c_3 e^{7t} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Ch 7 (c) $AP = PD$ $\quad D = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $\quad P = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}$

$$A = PDP^{-1} = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}^{-1} = \frac{1}{3} \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$$

Ch 7 (d) $\vec{u} = \vec{c}_1 e^{t} + \vec{c}_2 e^{2t}$ or $\vec{u} = c_1 e^{t} \vec{v}_1 + c_2 e^{2t} \vec{v}_2$

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Ch8. (Matrix Exponential) Complete all three problems.

☐ [40%] Ch8(a): Using any method in the lectures or the textbook, display the matrix exponential e^{At} for the 2×2 system. Then go on to solve the system $\mathbf{u}' = A\mathbf{u}$ for $\mathbf{u}$.

$$\begin{aligned} x' &= 2x, \\ y' &= y, \\ x(0) &= 1, \quad y(0) = 2. \end{aligned}$$

To save time, find e^{At} explicitly, because the answer is used in the next problem.

☐ [50%] Ch8(b): Display the matrix form of variation of parameters for the 2×2 system. Then integrate to find a particular solution.

$$\begin{aligned} x' &= 2x + 2, \\ y' &= y. \end{aligned}$$

☐ [10%] Ch8(c): Suppose $e^{At} = \begin{pmatrix} e^t & 0 \\ te^t & e^t \end{pmatrix}$. Find A, by using the matrix differential equation $\Phi'(t) = A\Phi(t)$.

Staple this page to the top of all Ch8 work. Submit one stapled package per chapter.

Ch8 (a) $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$, $e^{At} = \begin{pmatrix} e^{2t} & 0 \\ 0 & e^t \end{pmatrix}$

$$\vec{u}(t) = \begin{pmatrix} e^{2t} & 0 \\ 0 & e^t \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} e^{2t} \\ 2e^t \end{pmatrix}$$

Ch8 (b) $\vec{u}_p(t) = e^{At}\int_0^t e^{-As}\vec{F}(s)\,ds$

$$= e^{At}\int_0^t \begin{pmatrix} e^{-2s} & 0 \\ 0 & e^{-s} \end{pmatrix}\begin{pmatrix} 2 \\ 0 \end{pmatrix} ds$$

$$= e^{At}\int_0^t \begin{pmatrix} 2e^{-2s} \\ 0 \end{pmatrix} ds$$

$$= e^{At}\begin{pmatrix} 1-e^{-2t} \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} e^{2t} & 0 \\ 0 & e^t \end{pmatrix}\begin{pmatrix} 1-e^{-2t} \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} e^{2t}-1 \\ 0 \end{pmatrix}$$

Ch8 (c) $\Phi'(0) = A\,\Phi(0)$, $\Phi(t) = e^{At}$ satisfies $\Phi(0) = I$

$$\Rightarrow \quad A = \Phi'(0) = \begin{pmatrix} e^t & 0 \\ te^t+e^t & e^t \end{pmatrix}\Bigg|_{t=0} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

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Ch9. (Nonlinear Systems) Complete both problems.

☐ [30%] Ch9(a):
Determine whether the equilibrium $\mathbf{u} = \mathbf{0}$ is stable or unstable. Then classify the equilibrium point $\mathbf{u} = \mathbf{0}$ as a saddle, center, spiral or node.

$$\mathbf{u}' = \begin{pmatrix} 1 & 1 \\ -5 & -1 \end{pmatrix} \mathbf{u}$$

☐ [70%] Ch9(b):

(1) At the equilibrium point $x = 1$, $y = -2$ of the nonlinear system [see below], compute the matrix A of the linearized system $\mathbf{u}' = A\mathbf{u}$.

(2) Determine the stability at $t = \infty$ and the phase portrait classification for $\mathbf{u}' = A\mathbf{u}$ at $(0, 0)$.

(3) Apply a theorem to classify $x = 1$, $y = -2$ as a saddle, center, spiral or node for **the nonlinear system.**

$$\begin{aligned} x' &= y + 2x, \\ y' &= xy + 2. \end{aligned}$$

Staple this page to the top of all Ch9 work. Submit one stapled package per chapter.

Ch 9 (a) $A = \begin{pmatrix} 1 & 1 \\ -5 & -1 \end{pmatrix}$, $r^2 + 4 = 0$, $r = \pm 2i$, atoms = $\cos 2t$, $\sin 2t$

Stable center

Ch 9 (b) $J(x,y) = \begin{pmatrix} 2 & 1 \\ y & x \end{pmatrix}$, $J(1,-2) = \begin{pmatrix} 2 & 1 \\ -2 & 1 \end{pmatrix}$,

$r^2 - 3r + 4 = 0$, $r = \frac{3}{2} \pm \frac{1}{2}\sqrt{9-16} = \frac{3}{2} \pm \frac{1}{2}\sqrt{7}\,i$,

atoms = $e^{3t/2}\cos(\sqrt{7}\,t/2)$, $e^{3t/2}\sin(\sqrt{7}\,t/2)$.

(1) $A = \begin{pmatrix} 2 & 1 \\ -2 & 1 \end{pmatrix}$

(2) unstable at $t = \infty$

(3) unstable spiral for nonlinear system at $(1,-2)$

Name ________________ 7:30am Version 1

Differential Equations and Linear Algebra 2250-1 [7:30 class]

Final Exam at 8:00am on 5 May 2009

Ch10. (Laplace Transform Methods) Check the boxes ☐ on the **four problems** to be graded, which is 100%. Label worked problems accordingly.

It is assumed that you have memorized the basic 4-item Laplace integral table and know the 5 basic rules for Laplace integrals. No other tables or theory are required to solve the problems below. If you don't know a table entry, then leave the expression unevaluated for partial credit.

Ch10(a) [10%]: **No replacement. This problem is required.**

State the shifting rule and the convolution theorem.

Ch10(b) [25%]:**No replacement. This problem is required.**

Solve for $f(t)$ in the equation $\mathcal{L}(f(t)) = \dfrac{2s}{(s+1)^2(s-1)}$.

Ch10(c) [25%]: **No replacement. This problem is required.**

Solve for $f(t)$ in the equation $\dfrac{d}{ds}\mathcal{L}(f(t)) = \dfrac{2}{s^3} + \mathcal{L}(t^2 \sin t)$.

Ch10(d) [40%]:

Solve by Laplace's method for the solution $x(t)$:

$$x''(t) + 2x'(t) = 2e^{2t}, \quad x(0) = x'(0) = 0.$$

If you finished Ch10(a), Ch10(b), Ch10(c) and Ch10(d), then 100% has been marked -- the exam is finished. **Otherwise**, replace Ch10(d) with Ch10(e). **A maximum of four problems will be graded.**

Ch10(e) [40%]: **Ch10(d) will not be graded if you do this problem.**

Use Laplace's method to find an explicit formula for $y(t)$. Don't find $x(t)$!

$$\begin{aligned} x'(t) &= 2x(t) + 4y(t), \\ y'(t) &= 4x(t) + 2y(t), \\ x(0) &= 1, \\ y(0) &= 1. \end{aligned}$$

Staple this page to the top of all Ch10 work. Submit one stapled package per chapter.

For functions of exponential order:

Ch10 (a)

- $\mathcal{L}(e^{at} f(t)) = \mathcal{L}(f(t))\Big|_{s \to s-a}$
- $\mathcal{L}(f(t))\mathcal{L}(g(t)) = \mathcal{L}\left(\int_0^t f(t-u)g(u)\,du \right)$

Ch10 (b) $\mathcal{L}(f(t)) = \dfrac{2s}{(s+1)^2(s-1)} = \dfrac{a}{s+1} + \dfrac{b}{(s+1)^2} + \dfrac{c}{s-1}$

$= \mathcal{L}(ae^{-t} + bte^{-t} + ce^{t})$

$\boxed{f(t) = ae^{-t} + bte^{-t} + ce^{t} \quad a = -\tfrac{1}{2} \quad b = 1 \quad c = \tfrac{1}{2}}$

Partial fractions

clear fractions: $2s = a(s+1)(s-1) + b(s-1) + c(s+1)^2$

samples $\to$ $s = -1$: $-2 = 0 - 2b + 0$

$s = 1$: $2 = 0 + 0 + 4c$

$s = 0$: $0 = -a - b + c$

Ch10 (c) $\mathcal{L}((-t)f(t)) = \mathcal{L}(t^2) + \mathcal{L}(t^2 \sin t) \Rightarrow \boxed{f(t) = -t - t\sin t}$

Ch10 (d) $\mathcal{L}(x(t)) = \dfrac{2}{s(s+2)(s-2)} = -\tfrac{1}{2}\left(\tfrac{1}{s}\right) + \tfrac{1}{4}\left(\tfrac{1}{s+2}\right) + \tfrac{1}{4}\left(\tfrac{1}{s-2}\right)$

$\Rightarrow \boxed{x(t) = -\tfrac{1}{2} + \tfrac{1}{4}e^{-2t} + \tfrac{1}{4}e^{2t}}$

Ch10 (d) $A = \begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix}$. $\mathcal{L}(e^{At}) = (sI - A)^{-1} = \dfrac{1}{\Delta}\begin{pmatrix} s-2 & -4 \\ -4 & s-2 \end{pmatrix}$

where $\Delta = (s-2)^2 - 16 = s^2 - 4s - 12 = (s-6)(s+2)$

solve for both x, y

Then $\mathcal{L}(e^{At}) = \begin{pmatrix} \frac{s-2}{\Delta} & \frac{4}{\Delta} \\ \frac{4}{\Delta} & \frac{s-2}{\Delta} \end{pmatrix}$, $\dfrac{s-2}{(s-6)(s+2)} = \dfrac{1/2}{s-6} + \dfrac{1/2}{s+2}$

$\dfrac{4}{(s-6)(s+2)} = \dfrac{1/2}{s-6} + \dfrac{-1/2}{s+2}$

$e^{At} = \dfrac{1}{2}\begin{pmatrix} e^{6t} + e^{-2t} & e^{6t} - e^{-2t} \\ e^{6t} - e^{-2t} & e^{6t} + e^{-2t} \end{pmatrix}$

$\vec{u}(t) = e^{At}\vec{u}(0) = e^{At}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} e^{6t} \\ e^{6t} \end{pmatrix}$

$\boxed{x(t) = e^{6t} \quad y(t) = e^{6t}}$

Solve only for y(t)

- To solve for just $\mathcal{L}(y)$, multiply $(sI - A)^{-1}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ in the second position only to get $\mathcal{L}(y) = \dfrac{4 + s - 2}{\Delta} = \dfrac{1}{s-6} = \mathcal{L}(e^{6t}) \Rightarrow \boxed{y(t) = e^{6t}}$