Applied Differential Equations 2250
Midterm Exam 3, 10:45am

Exam date: Thursday, 23 April, 2009

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%.

1. **(ch5)** Complete any combination to make 100%.

(1a) [30%] A particular solution $y_p(x) = -x^4 - 12x^2$ is known for $\dfrac{d^4y}{dx^4} - \dfrac{d^2y}{dx^2} = 12x^2$. Find the general solution y.

(1b) [70%] Find the steady-state periodic solution for the forced spring-mass system $x'' + 2x' + 10x = 170\sin(t)$.

(1c) [15%] Determine the practical resonance frequency ω for the electric current equation $I'' + 7I' + 100I = 50\omega\cos(\omega t)$.

Use this page to start your solution. Attach extra pages as needed, then staple.

2. **(ch5)** Complete enough parts to make 100%.

(2a) [70%] A homogeneous linear differential equation with constant coefficients has characteristic equation of order 6 with roots $0, 0$, $1, 1$, $3i$, $-3i$, listed according to multiplicity. We replace the 0 on the right side by $f(x) = 3xe^{-x} + 4x^2 + 5\sin 3x$ to obtain a non-homogeneous differential equation in variables x, y. Determine the undetermined coefficients **shortest** trial solution for y_p. To save time, **do not** evaluate the undetermined coefficients and **do not** find $y_p(x)$! Undocumented detail or guessing earns no credit.

(2b) [30%] The general solution of a certain linear homogeneous differential equation with constant coefficients is

$$y = c_1 e^x + c_2 x e^x + c_3 + c_4 x + c_5 e^{-x}.$$

Find the factored form of the characteristic polynomial.

(2c) [15%] Find five independent solutions of a homogeneous linear constant coefficient differential equation whose sixth order characteristic equation has roots 1, 1, 1, 0, $1 + i$, $1 - i$.

(2d) [30%] Let $f(x) = 4xe^x \sin x$. Find a constant-coefficient linear homogeneous differential equation of smallest order which has $f(x)$ as a solution.

Use this page to start your solution. Attach extra pages as needed, then staple.

3. **(ch10)** Complete enough parts to make 100%. It is assumed that you have memorized the basic 4-item Laplace integral table and know the 6 basic rules for Laplace integrals. No other tables or theory are required to solve the problems below. If you don't know a table entry, then leave the expression unevaluated for partial credit.

(3a) [75%] Display the details of Laplace's method to solve the system for $y(t)$. Don't waste time solving for $x(t)$!

The answer is $y(t) = e^t - te^t$ – **solution details are required**.

Suggestion: Save effort by using the Laplace resolvent equation $(sI - A)\mathcal{L}(\mathbf{u}) = \mathbf{u}(0)$ and Cramer's Rule. Notation: $\mathbf{u}$ is the vector solution of $\mathbf{u}' = A\mathbf{u}$ with components $x(t)$, $y(t)$.

$$\begin{aligned} &x' = 2x - y, \\ &y' = x, \\ &x(0) = 0, \quad y(0) = 1. \end{aligned}$$

(3b) [25%] Find $f(t)$ by partial fraction methods, given

$$\mathcal{L}(f(t)) = \frac{2s+5}{s(s+4)}.$$

(3c) [25%] Solve for $f(t)$, given

$$\frac{d}{ds}\mathcal{L}(f(t)) = \frac{2}{s^3} + \frac{d}{ds}\left(\frac{3s}{s^2+1}\right).$$

(3d) [25%] Solve for $f(t)$, given

$$\mathcal{L}(e^{-3t}f(t)) = \frac{s+2}{(s+3)^2}$$

Use this page to start your solution. Attach extra pages as needed, then staple.

4. **(ch10)** Complete both parts.

(4a) [50%] Solve by Laplace's method for the solution $x(t)$:

$$x''(t) + x(t) = 4e^t, \quad x(0) = x'(0) = 0.$$

(4b) [50%] Apply Laplace's method to find a formula for $\mathcal{L}(x(t))$. To save time, **do not** solve for $x(t)$! Document steps by reference to tables and rules.

$$x^{(4)} + x^{(2)} = 3t^2 + 4e^t + 5\sin 2t, \quad x(0) = x'(0) = x''(0) = 0, \quad x'''(0) = -1.$$

Use this page to start your solution. Attach extra pages as needed, then staple.

5. **(ch6)** Complete enough parts to make 100%.

(5a) [20%] Find the eigenvalues of the matrix $A = \begin{pmatrix} -2 & 5 & 1 & 12 \\ -1 & 4 & -3 & 15 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & -1 & 4 \end{pmatrix}$. To save time, **do not** find eigenvectors!

(5b) [40%] Given $A = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, which has eigenvalues $1, 1, -1$, assume there exists an invertible matrix P and a diagonal matrix D such that $AP = PD$. Circle those vectors from the list below which are possible columns of P.

$$\begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.$$

(5c) [20%] Suppose a 2×2 matrix A has eigenpairs $\left(3, \begin{pmatrix} 1 \\ 3 \end{pmatrix}\right), \left(-2, \begin{pmatrix} 1 \\ -2 \end{pmatrix}\right)$. Display an invertible matrix P and a diagonal matrix D such that $AP = PD$.

(5d) [20%] Let $P = \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix}$, $D = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}$ and define A by $AP = PD$. Display the eigenpairs of A.

(5e) [20%] Assume the vector general solution $\mathbf{x}(t)$ of the linear differential system $\mathbf{x}' = C\mathbf{x}$ is given by

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Display an invertible matrix P and a diagonal matrix D such that $CP = PD$.

Use this page to start your solution. Attach extra pages as needed, then staple.