Name ____________________________ 2250 Midterm 3 [7:30, S2009]

Applied Differential Equations 2250
Midterm Exam 3, 7:30am

Exam date: Thursday, 23 April, 2009

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%.

1. **(ch5)** Complete any combination to make 100%.

(1a) [70%] Find the steady-state periodic solution for the forced spring-mass system $x'' + 2x' + 10x = 170\sin(t)$.

Answer: $x = 18\sin t - 4\cos t$

(1b) [70%] Write the solution of $x'' + 4x = 6\sin t$, $x(0) = x'(0) = 0$, as the sum of two harmonic oscillations of different natural frequencies. **To save time, don't convert to phase-amplitude form**.

Answer: $x(t) = 2\sin t - \sin 2t$

(1c) [30%] Determine the practical resonance frequency ω for the spring-mass system equation $x'' + 2x' + 5x = 50\cos(\omega t)$.

Answer: $\omega = \sqrt{k/m - c^2/(2m^2)} = \sqrt{5 - 4/2} = \sqrt{3}$.

Use this page to start your solution. Attach extra pages as needed, then staple.

2. **(ch5)** Complete enough parts to make 100%.

(2a) [70%] A homogeneous linear differential equation with constant coefficients has characteristic equation of order 6 with roots $0, 0,\ -1, -1,\ 2i, -2i$, listed according to multiplicity. We replace the 0 on the right side by $f(x) = 3xe^{-x} + 4x^3 + 5\sin 3x$ to obtain a non-homogeneous differential equation in variables x, y. Determine the undetermined coefficients **shortest** trial solution for y_p. To save time, **do not** evaluate the undetermined coefficients and **do not** find $y_p(x)$! Undocumented detail or guessing earns no credit.

Answer: The trial solution is a linear combination of the 8 atoms $x^2e^{-x}, x^3e^{-x}, \quad x^2, x^3, x^4, x^5, \quad \cos 3x, \sin 3x$.

(2b) [30%] The general solution of a certain linear homogeneous differential equation with constant coefficients is

$$y = c_1e^{2x} + c_2xe^{2x} + c_3 + c_4x + c_5e^{-x}.$$

Find the factored form of the characteristic polynomial.

Answer: The atoms are constructed from roots $2, 2, 0, 0, -1$, listed according to multiplicity. Then $(r-2)^2$, r^2 and $(r+1)$ are factors. The characteristic polynomial is $a(r-2)^2r^2(r+1)$ for some nonzero constant a.

(2c) [30%] Let $f(x) = 4xe^x \sin x$. Find a constant-coefficient linear homogeneous differential equation of smallest order which has $f(x)$ as a solution.

Answer: The atom $xe^x \sin x$ is constructed from roots $1 + i, 1 + i, 1 - i, 1 - i$, listed according to multiplicity. Then the factors of the characteristic polynomial must include $(r-1)^2 + 1$ and $((r-1)^2 + 1)^2$. The characteristic polynomial must be a constant multiple of $((r-1)^2 + 1)^2 = r^4 - 4r^3 + 8r^2 - 8r + 4$. This characteristic equation belongs to the differential equation $y^{(4)} - 4y^{(3)} + 8y^{(2)} - 8y' + 4y = 0$.

Use this page to start your solution. Attach extra pages as needed, then staple.

3. **(ch10)** Complete enough parts to make 100%. It is assumed that you have memorized the basic 4-item Laplace integral table and know the 6 basic rules for Laplace integrals. No other tables or theory are required to solve the problems below. If you don't know a table entry, then leave the expression unevaluated for partial credit.

(3a) [75%] Display the details of Laplace's method to solve the system for $y(t) = e^t - te^t$. Don't waste time solving for $x(t)$!

Suggestion: Save effort by using the Laplace resolvent equation $(sI - A)\mathcal{L}(\mathbf{u}) = \mathbf{u}(0)$ and Cramer's Rule. Notation: $\mathbf{u}$ is the vector solution of $\mathbf{u}' = A\mathbf{u}$ with components $x(t)$, $y(t)$.

$$\begin{aligned} x' &= 2x - y, \\ y' &= x, \\ x(0) &= 0, \quad y(0) = 1. \end{aligned}$$

Answer: $x(t) = -te^t$, $y(t) = e^t - te^t$.

(3b) [25%] Find $f(t)$ by partial fraction methods, given

$$\mathcal{L}(f(t)) = \frac{2s+5}{s(s-4)}.$$

Answer: $\mathcal{L}(f(t)) = \frac{-5/4}{s} + \frac{13/4}{s-4} = \mathcal{L}(-5/4 + (13/4)e^{4t})$

(3c) [25%] Solve for $f(t)$, given

$$\frac{d}{ds}\mathcal{L}(f(t)) = \frac{1}{s^2} + \frac{3}{s^2+1}.$$

Answer: $-tf(t) = t + 3\sin t$ or $f(t) = -1 - \frac{3\sin t}{t}$.

(3d) [25%] Solve for $f(t)$, given

$$\mathcal{L}(e^{-3t}f(t)) = \frac{s+1}{(s+2)^2}$$

Answer: $\mathcal{L}(e^{-3t}f(t)) = \left.\frac{s-1}{s^2}\right|_{s\to(s+2)} = \left.\left(\frac{1}{s} - \frac{1}{s^2}\right)\right|_{s\to(s+2)} = \mathcal{L}((1-t)e^{-2t})$. Then $f(t) = (1-t)e^{-2t}e^{3t} = (1-t)e^t$.

Use this page to start your solution. Attach extra pages as needed, then staple.

4. **(ch10)** Complete enough parts to make 100%.

(4a) [50%] Fill in the blank spaces in the Laplace table:

$f(t)$	t^3			$e^{-t}\sin t$	$2te^t$
$\mathcal{L}(f(t))$	$\dfrac{6}{s^4}$	$\dfrac{1}{2s+3}$	$\dfrac{s+1}{s^2+2s+5}$		

Answer: Left to right: $\frac{1}{2}e^{-3t/2}$, $e^{-t}\cos 2t$, $\frac{1}{(s+1)^2+1}$, $\frac{2}{(s-1)^2}$.

(4b) [50%] Solve by Laplace's method for the solution $x(t)$:

$$x''(t) + x(t) = 2e^t, \quad x(0) = x'(0) = 0.$$

Answer: $x(t) = -\sin(t) - \cos(t) + e^t$.

(4c) [50%] Apply Laplace's method to find a formula for $\mathcal{L}(x(t))$. To save time, **do not** solve for $x(t)$! Document steps by reference to tables and rules.

$$x^{(4)} + x^{(2)} = 3t + 4e^t + 5\sin 2t, \quad x(0) = x'(0) = x''(0) = 0, \quad x'''(0) = -1.$$

Answer: $\mathcal{L}(x(t)) = p/q$, $p = -1 + \mathcal{L}(\text{RHS})$, $q = s^4 + s^2$. Finally $p = -1 + \mathcal{L}(3t + 4e^t + 5\sin 2t) = -1 + \frac{3}{s^2} + \frac{4}{s-1} + \frac{10}{s^2+4}$. Rules: Parts $\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$, Linearity. Tables used to evaluate $\mathcal{L}(\text{RHS})$.

Use this page to start your solution. Attach extra pages as needed, then staple.

5. **(ch6)** Complete enough parts to make 100%.

(5a) [20%] Find the eigenvalues of the matrix $A = \begin{pmatrix} -2 & 7 & 1 & 12 \\ -1 & 6 & -3 & 15 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$. To save time, **do not** find eigenvectors!

Answer: $-1, 5, 2 \pm i$

(5b) [40%] Given $A = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, which has eigenvalues $1, 1, -1$, assume there exists an invertible matrix P and a diagonal matrix D such that $AP = PD$. Circle those vectors from the list below which are possible columns of P.

$$\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.$$

Answer: Matrix P must contain eigenvectors of P corresponding to eigenvalues 1, 1, -1, in some order. For each given vector $\mathbf{v}$, multiply $A\mathbf{v}$ and see if it is $\lambda\mathbf{v}$ for some λ. The first fails. The second works for $\lambda = 1$. The third fails.

(5c) [20%] Suppose a 2×2 matrix A has eigenpairs $\left(2, \begin{pmatrix} 1 \\ 3 \end{pmatrix}\right), \left(-2, \begin{pmatrix} 1 \\ -2 \end{pmatrix}\right)$. Display an invertible matrix P and a diagonal matrix D such that $AP = PD$.

Answer: Define $P = \begin{pmatrix} 1 & 1 \\ 3 & -2 \end{pmatrix}$, $D = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$. Then $AP = PD$.

(5d) [20%] Let $P = \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix}$, $D = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}$ and define A by $AP = PD$. Display the eigenpairs of A.

Answer: $\left(3, \begin{pmatrix} 3 \\ 1 \end{pmatrix}\right), \left(-2, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right)$

(5e) [20%] Assume the vector general solution $\mathbf{x}(t)$ of the linear differential system $\mathbf{x}' = C\mathbf{x}$ is given by

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^t \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Display an invertible matrix P and a diagonal matrix D such that $CP = PD$.

Answer: The missing exponential in the first term is e^{0t}. The eigenvalues come from the coefficients in the exponentials, 0 and 1. The eigenpairs are $\left(0, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right), \left(1, \begin{pmatrix} -1 \\ 1 \end{pmatrix}\right)$. Then $P = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

Use this page to start your solution. Attach extra pages as needed, then staple.