Name ___________________________ Sample 2250 Midterm 3 [7:30+10:45, S2009]

Applied Differential Equations 2250
Sample Midterm Exam 3

Exam date: Thursday, 23 April, 2009

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%.

1. **(ch5)** Complete any combination to make 100%. in order to give you an idea of the range of possible questions.

(1a) [60%] Find a particular solution $y_p(x)$ for $\dfrac{d^4y}{dx^4} + 4\dfrac{d^2y}{dx^2} = 24x$.

(1b) [60%] Find the steady-state periodic solution for the forced spring-mass system $x'' + 2x' + 5x = 10\sin(t)$.

(1c) [40%] Given $4x''(t) + 10x'(t) + 2wx(t) = 0$, which represents a damped spring-mass system with $m = 4$, $c = 10$, $k = 2w$, determine all values of w such that the equation is under-damped. **Do not solve for** $x(t)$!

(1d) [40%] Find by variation of parameters an integral formula for a particular solution x_p of the equation $x'' + 4x' + 20x = e^{t^2}\ln(t^2 + 1)$. To save time, don't try to evaluate integrals (it's impossible).

(1e) [40%] Write the solution of $x'' + 9x = 30\sin t$, $x(0) = x'(0) = 0$, as the sum of two harmonic oscillations of different natural frequencies. **To save time, don't convert to phase-amplitude form**.

(1f) [30%] Determine the practical resonance frequency ω for the current equation $I'' + 2I' + 10I = 100\omega\cos(\omega t)$. [original had a typo]

(1g) [30%] Determine the practical resonance frequency ω for the spring-mass system equation $2x'' + 4x'' + 10x = 100\cos(\omega t)$.

2. **(ch5)** Complete enough parts to make 100%.

(2a) [40%] A non-homogeneous linear differential equation with constant coefficients has right side $f(x) = x^2e^{-x} + x^3(x + 4) + e^x\cos 2x$ and characteristic equation of order 8 with roots $0, 0, 0,\ 1,\ -1, -1,\ 2i, -2i$, listed according to multiplicity. Determine the undetermined coefficients **shortest** trial solution for y_p. To save time, **do not** evaluate the undetermined coefficients and **do not** find $y_p(x)$! Undocumented detail or guessing earns no credit.

(2b) [30%] Find the corrected trial solution in the method of undetermined coefficients for the differential equation $y'' + y = 3\cos x$. To save time, **do not** evaluate the undetermined coefficients and **do not** find $y_p(x)$!

(2c) [50%] Assume $f(x)$ is a solution of a constant-coefficient linear homogeneous differential equation whose factored characteristic equation is $r^2(r + 1)(r^2 + 9) = 0$. Find the shortest trial solution in the method of undetermined coefficients for the differential equation $y''' - y' = f(x)$. To save time, **do not** evaluate the undetermined coefficients and **do not** find $y_p(x)$!

(2d) [50%] Find the shortest trial solution in the method of undetermined coefficients for the differential equation $y''' - y' = x + e^x$. To save time, **do not** evaluate the undetermined coefficients and **do not** find $y_p(x)$!

(2e) [25%] The general solution of a linear homogeneous differential equation with constant coefficients is

$$y = c_1e^x\cos 2x + c_2e^x\sin 2x + c_3 + c_4x + c_5x^2.$$

Find the factored form of the characteristic polynomial.

(2f) [25%] Find six independent solutions of the homogeneous linear constant coefficient differential equation whose sixth order characteristic equation has roots -1, -1, 0, 0, $2+i$, $2-i$.

(2g) [25%] Let $f(x) = x^3 + x \sin 2x$. Find a constant-coefficient linear homogeneous differential equation which has $f(x)$ as a solution.

(2h) [20%] Let $f(x) = 4e^x - \cosh x + e^x \cos^2 2x$. Find the characteristic polynomial of a constant-coefficient linear homogeneous differential equation which has $f(x)$ as a solution.

3. **(ch10)** Complete enough parts to make 100%. It is assumed that you have memorized the basic 4-item Laplace integral table and know the 6 basic rules for Laplace integrals. No other tables or theory are required to solve the problems below. If you don't know a table entry, then leave the expression unevaluated for partial credit.

(3a) [50%] Apply Laplace's method to solve the system for $x(t)$. Don't waste time solving for $y(t)$!

$$\begin{aligned} x' &= 3y, \\ y' &= 2x - y, \\ x(0) &= 0, \quad y(0) = 1. \end{aligned}$$

(3b) [25%] Find $f(t)$ by partial fraction methods, given

$$\mathcal{L}(f(t)) = \frac{3s+4}{s(s+4)}.$$

(3c) [25%] Solve for $f(t)$, given

$$\frac{d}{ds}\mathcal{L}(f(t)) = \mathcal{L}\left(te^{2t}\right)\Big|_{s \to (s+1)}$$

(3d) [25%] Solve for $f(t)$, given

$$\mathcal{L}(f(t)) = \left(\frac{s+2}{s+1}\right)^2 \frac{1}{s+1}$$

4. **(ch10)** Complete enough parts to make 100%.

(4a) [25%] Fill in the blank spaces in the Laplace table:

$f(t)$	t^3			$e^t \cos t$	$t^2 e^{-t}$
$\mathcal{L}(f(t))$	$\frac{6}{s^4}$	$\frac{1}{2s+2}$	$\frac{s+1}{s^2+2s+10}$		

Answers: Left to right: $\frac{1}{2}e^{-t}$, $e^{-t}\cos 3t$, $\frac{s-1}{(s-1)^2+1}$, $\frac{2}{(s+1)^3}$.

(4b) [50%] Solve by Laplace's method for the solution $x(t)$:

$$x''(t) + x'(t) = 9e^{-t}, \quad x(0) = x'(0) = 0.$$

(4c) [25%] Solve for $x(t)$, given

$$\mathcal{L}(x(t)) = \frac{d}{ds}\left(\mathcal{L}(e^{2t}\sin 2t)\right) + \mathcal{L}(t\sin t)\big|_{s\to(s+2)}.$$

(4d) [25%] Solve for $x(t)$, given

$$\mathcal{L}(x(t)) = \frac{s+2}{(s+1)^2} + \frac{1+s}{s^2+5s}$$

(4e) [25%] Find $\mathcal{L}(f(t))$, given $f(t) = e^{2t}\left(\frac{\sin(t)}{t}\right)$.

(4f) [30%] Apply Laplace's method to find a formula for $\mathcal{L}(x(t))$. **Do not** solve for $x(t)$! Document steps by reference to tables and rules.

$$\frac{d^4x}{dt^4} + 4\frac{d^2x}{dt^2} = e^t(5t + 4e^t + 3\sin 3t), \quad x(0) = x'(0) = x''(0) = 0, \quad x'''(0) = -1.$$

(4g) [25%] Find $\mathcal{L}(f(t))$, given $f(t) = u(t-\pi)\frac{\sin(t)}{t-\pi}$, where u is the unit step function [original had a typo].

(4h) [40%] Find $f(t)$ by partial fraction methods, given

$$\mathcal{L}(f(t)) = \frac{8s^2 - 24}{(s-1)(s+3)(s+1)^2}.$$

(4i) [30%] Apply Laplace's method to find a formula for $\mathcal{L}(x(t))$. To save time, **do not** solve for $x(t)$! Document steps by reference to tables and rules.

$$x^{(4)} - x^{(2)} = 3t^2 + 4e^{-2t} + 5e^t\sin 2t, \quad x(0) = x'(0) = x''(0) = 0, \quad x'''(0) = -1.$$

5. **(ch6)** Complete all of the items below.

(5a) [30%] Find the eigenvalues of the matrix $A = \begin{pmatrix} -2 & 7 & 1 & 12 \\ -1 & 6 & -3 & 15 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & -2 & 3 \end{pmatrix}$. To save time, **do not** find eigenvectors!

(5b) [40%] Given $A = \begin{pmatrix} 5 & 1 & -1 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix}$, assume there exists an invertible matrix P and a diagonal matrix D such that $AP = PD$. Circle all possible columns of P from the list below.

$$\begin{pmatrix} 2 \\ -3 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

(5c) [30%] Find all eigenpairs for the matrix $A = \begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix}$. Then display Fourier's model for A.

(5d) [50%] Find the remaining eigenpairs of

$$E = \begin{pmatrix} 6 & 2 & -2 \\ 0 & 5 & 1 \\ 0 & 1 & 5 \end{pmatrix}$$

provided we already know one eigenpair

$$\left(6, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right).$$

(5e) [25%] Suppose a 2×2 matrix A has eigenpairs $\left(3, \begin{pmatrix} 3 \\ 1 \end{pmatrix}\right), \left(-2, \begin{pmatrix} 1 \\ -2 \end{pmatrix}\right)$. Display an invertible matrix P and a diagonal matrix D such that $AP = PD$.

(5f) [25%] Assume the vector general solution $\mathbf{x}(t)$ of the linear differential system $\mathbf{x}' = A\mathbf{x}$ is given by

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} + c_3 e^{2t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Display Fourier's model for the 3×3 matrix A.

(5g) [30%] Find the eigenvalues of the matrix $A = \begin{pmatrix} -2 & 7 & 1 & 27 \\ -1 & 6 & -3 & 62 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & -1 & 0 \end{pmatrix}$. To save time, **do not** find eigenvectors!

(5h) [30%] Assume A is 2×2 and Fourier's model holds:

$$A\left(c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right) = 2c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Find A.

(5i) [40%] Let $A = \begin{pmatrix} 3 & 0 & -1 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$. Circle the possible eigenvectors of A in the list below.

$$\begin{pmatrix} -4 \\ 2 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

(5j) [40%] Consider the 3×3 matrix

$$E = \begin{pmatrix} 4 & 2 & -2 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix}.$$

Show that matrix E has a Fourier model: [original had a typo]

$$E\left(c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}\right) = 4c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 4c_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + 2c_3 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}.$$

(5k) [40%] Define E as in the previous problem. Find $E^3 \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$ without using matrix multiply.

(5l) [30%] Let $P = \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix}$, $D = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}$ and define A by $AP = PD$. Display the eigenpairs of A.

(5m) [30%] Assume the vector general solution $\mathbf{x}(t)$ of the linear differential system $\mathbf{x}' = M\mathbf{x}$ is given by

$$\mathbf{x}(t) = c_1 e^{-t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Display an invertible matrix P and a diagonal matrix D such that $MP = PD$.