

## Applied Differential Equations 2250

### Sample Midterm Exam 3

Exam date: Thursday, 23 April, 2009

**Instructions:** This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%.

1. (ch5) Complete any combination to make 100%. in order to give you an idea of the range of possible questions.

(1a) [60%] Find a particular solution  $y_p(x)$  for  $\frac{d^4y}{dx^4} + 4\frac{d^2y}{dx^2} = 24x$ .

(1b) [60%] Find the steady-state periodic solution for the forced spring-mass system  $x'' + 2x' + 5x = 10 \sin(t)$ .

(1c) [40%] Given  $4x''(t) + 10x'(t) + 2wx(t) = 0$ , which represents a damped spring-mass system with  $m = 4$ ,  $c = 10$ ,  $k = 2w$ , determine all values of  $w$  such that the equation is under-damped. **Do not solve for  $x(t)$ !**

(1d) [40%] Find by variation of parameters an integral formula for a particular solution  $x_p$  of the equation  $x'' + 4x' + 20x = e^{t^2} \ln(t^2 + 1)$ . To save time, don't try to evaluate integrals (it's impossible).

(1e) [40%] Write the solution of  $x'' + 9x = 30 \sin t$ ,  $x(0) = x'(0) = 0$ , as the sum of two harmonic oscillations of different natural frequencies. **To save time, don't convert to phase-amplitude form.**

(1f) [30%] Determine the practical resonance frequency  $\omega$  for the current equation  $I'' + 2I' + 10I = 100\omega \cos(\omega t)$ . [original had a typo]

(1g) [30%] Determine the practical resonance frequency  $\omega$  for the spring-mass system equation  $2x'' + 4x'' + 10x = 100 \cos(\omega t)$ .

2. (ch5) Complete enough parts to make 100%.

(2a) [40%] A non-homogeneous linear differential equation with constant coefficients has right side  $f(x) = x^2 e^{-x} + x^3(x+4) + e^x \cos 2x$  and characteristic equation of order 8 with roots 0, 0, 0, 1, -1, -1,  $2i$ ,  $-2i$ , listed according to multiplicity. Determine the undetermined coefficients **shortest** trial solution for  $y_p$ . To save time, **do not** evaluate the undetermined coefficients and **do not** find  $y_p(x)$ ! Undocumented detail or guessing earns no credit.

(2b) [30%] Find the corrected trial solution in the method of undetermined coefficients for the differential equation  $y'' + y = 3 \cos x$ . To save time, **do not** evaluate the undetermined coefficients and **do not** find  $y_p(x)$ !

(2c) [50%] Assume  $f(x)$  is a solution of a constant-coefficient linear homogeneous differential equation whose factored characteristic equation is  $r^2(r+1)(r^2+9) = 0$ . Find the shortest trial solution in the method of undetermined coefficients for the differential equation  $y''' - y' = f(x)$ . To save time, **do not** evaluate the undetermined coefficients and **do not** find  $y_p(x)$ !

(2d) [50%] Find the shortest trial solution in the method of undetermined coefficients for the differential equation  $y''' - y' = x + e^x$ . To save time, **do not** evaluate the undetermined coefficients and **do not** find  $y_p(x)$ !

(2e) [25%] The general solution of a linear homogeneous differential equation with constant coefficients is

$$y = c_1 e^x \cos 2x + c_2 e^x \sin 2x + c_3 + c_4 x + c_5 x^2.$$

Find the factored form of the characteristic polynomial.

(2f) [25%] Find six independent solutions of the homogeneous linear constant coefficient differential equation whose sixth order characteristic equation has roots  $-1, -1, 0, 0, 2 + i, 2 - i$ .

(2g) [25%] Let  $f(x) = x^3 + x \sin 2x$ . Find a constant-coefficient linear homogeneous differential equation which has  $f(x)$  as a solution.

(2h) [20%] Let  $f(x) = 4e^x - \cosh x + e^x \cos^2 2x$ . Find the characteristic polynomial of a constant-coefficient linear homogeneous differential equation which has  $f(x)$  as a solution.

3. (ch10) Complete enough parts to make 100%. It is assumed that you have memorized the basic 4-item Laplace integral table and know the 6 basic rules for Laplace integrals. No other tables or theory are required to solve the problems below. If you don't know a table entry, then leave the expression unevaluated for partial credit.

(3a) [50%] Apply Laplace's method to solve the system for  $x(t)$ . Don't waste time solving for  $y(t)$ !

$$\begin{aligned} x' &= 3y, \\ y' &= 2x - y, \\ x(0) &= 0, \quad y(0) = 1. \end{aligned}$$

(3b) [25%] Find  $f(t)$  by partial fraction methods, given

$$\mathcal{L}(f(t)) = \frac{3s + 4}{s(s + 4)}.$$

(3c) [25%] Solve for  $f(t)$ , given

$$\frac{d}{ds} \mathcal{L}(f(t)) = \mathcal{L}(te^{2t}) \Big|_{s \rightarrow (s+1)}$$

(3d) [25%] Solve for  $f(t)$ , given

$$\mathcal{L}(f(t)) = \left( \frac{s+2}{s+1} \right)^2 \frac{1}{s+1}$$

4. (ch10) Complete enough parts to make 100%.

(4a) [25%] Fill in the blank spaces in the Laplace table:

$f(t)$	$t^3$			$e^t \cos t$	$t^2 e^{-t}$
$\mathcal{L}(f(t))$	$\frac{6}{s^4}$	$\frac{1}{2s+2}$	$\frac{s+1}{s^2+2s+10}$		

Answers: Left to right:  $\frac{1}{2}e^{-t}$ ,  $e^{-t} \cos 3t$ ,  $\frac{s-1}{(s-1)^2+1}$ ,  $\frac{2}{(s+1)^3}$ .

(4b) [50%] Solve by Laplace's method for the solution  $x(t)$ :

$$x''(t) + x'(t) = 9e^{-t}, \quad x(0) = x'(0) = 0.$$

(4c) [25%] Solve for  $x(t)$ , given

$$\mathcal{L}(x(t)) = \frac{d}{ds} \left( \mathcal{L}(e^{2t} \sin 2t) \right) + \mathcal{L}(t \sin t)|_{s \rightarrow (s+2)}.$$

(4d) [25%] Solve for  $x(t)$ , given

$$\mathcal{L}(x(t)) = \frac{s+2}{(s+1)^2} + \frac{1+s}{s^2+5s}$$

(4e) [25%] Find  $\mathcal{L}(f(t))$ , given  $f(t) = e^{2t} \left( \frac{\sin(t)}{t} \right)$ .

(4f) [30%] Apply Laplace's method to find a formula for  $\mathcal{L}(x(t))$ . **Do not** solve for  $x(t)$ ! Document steps by reference to tables and rules.

$$\frac{d^4 x}{dt^4} + 4 \frac{d^2 x}{dt^2} = e^t(5t + 4e^t + 3 \sin 3t), \quad x(0) = x'(0) = x''(0) = 0, \quad x'''(0) = -1.$$

(4g) [25%] Find  $\mathcal{L}(f(t))$ , given  $f(t) = u(t - \pi) \frac{\sin(t)}{t - \pi}$ , where  $u$  is the unit step function [original had a typo].

(4h) [40%] Find  $f(t)$  by partial fraction methods, given

$$\mathcal{L}(f(t)) = \frac{8s^2 - 24}{(s-1)(s+3)(s+1)^2}.$$

(4i) [30%] Apply Laplace's method to find a formula for  $\mathcal{L}(x(t))$ . To save time, **do not** solve for  $x(t)$ ! Document steps by reference to tables and rules.

$$x^{(4)} - x^{(2)} = 3t^2 + 4e^{-2t} + 5e^t \sin 2t, \quad x(0) = x'(0) = x''(0) = 0, \quad x'''(0) = -1.$$

5. (ch6) Complete all of the items below.

(5a) [30%] Find the eigenvalues of the matrix  $A = \begin{pmatrix} -2 & 7 & 1 & 12 \\ -1 & 6 & -3 & 15 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & -2 & 3 \end{pmatrix}$ . To save time, **do not** find eigenvectors!

(5b) [40%] Given  $A = \begin{pmatrix} 5 & 1 & -1 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix}$ , assume there exists an invertible matrix  $P$  and a diagonal matrix  $D$  such that  $AP = PD$ . Circle all possible columns of  $P$  from the list below.

$$\begin{pmatrix} 2 \\ -3 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

(5c) [30%] Find all eigenpairs for the matrix  $A = \begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix}$ . Then display Fourier's model for  $A$ .

(5d) [50%] Find the remaining eigenpairs of

$$E = \begin{pmatrix} 6 & 2 & -2 \\ 0 & 5 & 1 \\ 0 & 1 & 5 \end{pmatrix}$$

provided we already know one eigenpair

$$\left( 6, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right).$$

(5e) [25%] Suppose a  $2 \times 2$  matrix  $A$  has eigenpairs  $\left( 3, \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right), \left( -2, \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right)$ . Display an invertible matrix  $P$  and a diagonal matrix  $D$  such that  $AP = PD$ .

(5f) [25%] Assume the vector general solution  $\mathbf{x}(t)$  of the linear differential system  $\mathbf{x}' = A\mathbf{x}$  is given by

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} + c_3 e^{2t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Display Fourier's model for the  $3 \times 3$  matrix  $A$ .

(5g) [30%] Find the eigenvalues of the matrix  $A = \begin{pmatrix} -2 & 7 & 1 & 27 \\ -1 & 6 & -3 & 62 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & -1 & 0 \end{pmatrix}$ . To save time, **do not** find

eigenvectors!

(5h) [30%] Assume  $A$  is  $2 \times 2$  and Fourier's model holds:

$$A \left( c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right) = 2c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Find  $A$ .

(5i) [40%] Let  $A = \begin{pmatrix} 3 & 0 & -1 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ . Circle the possible eigenvectors of  $A$  in the list below.

$$\begin{pmatrix} -4 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

(5j) [40%] Consider the  $3 \times 3$  matrix

$$E = \begin{pmatrix} 4 & 2 & -2 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix}.$$

Show that matrix  $E$  has a Fourier model: [original had a typo]

$$E \left( c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \right) = 4c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 4c_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + 2c_3 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}.$$

(5k) [40%] Define  $E$  as in the previous problem. Find  $E^3 \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$  without using matrix multiply.

(5l) [30%] Let  $P = \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix}$ ,  $D = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}$  and define  $A$  by  $AP = PD$ . Display the eigenpairs of  $A$ .

(5m) [30%] Assume the vector general solution  $\mathbf{x}(t)$  of the linear differential system  $\mathbf{x}' = M\mathbf{x}$  is given by

$$\mathbf{x}(t) = c_1 e^{-t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Display an invertible matrix  $P$  and a diagonal matrix  $D$  such that  $MP = PD$ .