Applied Differential Equations 2250
Sample Midterm Exam 3

Exam date: Thursday, 23 April, 2009

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%.

1. **(ch5)** Complete any combination to make 100%. in order to give you an idea of the range of possible questions.

(1a) [60%] Find a particular solution $y_p(x)$ for $\dfrac{d^4y}{dx^4} + 4\dfrac{d^2y}{dx^2} = 24x$.

Answer: $y = x^3$

(1b) [60%] Find the steady-state periodic solution for the forced spring-mass system $x'' + 2x' + 5x = 10\sin(t)$.

Answer: $x = 2\sin t - \cos t$

(1c) [40%] Given $4x''(t) + 10x'(t) + 2wx(t) = 0$, which represents a damped spring-mass system with $m = 4$, $c = 10$, $k = 2w$, determine all values of w such that the equation is under-damped. **Do not solve for $x(t)$!**

Answer: The discriminant is $25 - 8w$. This must be negative to produce a complex root, therefore *under-damped* happens exactly when $w > 25/8$.

(1d) [40%] Find by variation of parameters an integral formula for a particular solution x_p of the equation $x'' + 4x' + 20x = e^{t^2}\ln(t^2+1)$. To save time, don't try to evaluate integrals (it's impossible).

Answer: $x_p(t) = x_1(t)\int_0^t k1(u)f(u)du + x_2(t)\int_0^t k2(u)f(u)du$, $f(t) = e^{t^2}\ln(t^2+1)$, $x_1(u) = e^{-2u}\cos(4u)$, $x_2(u) = \frac{1}{2}e^{-2u}\sin(4u)$, $k_1(u) = -x_2(u)/W(u)$, $k_2(u) = x_1(u)/W(u)$, $W(u) = 2e^{-4u}$.

(1e) [40%] Write the solution of $x'' + 9x = 30\sin t$, $x(0) = x'(0) = 0$, as the sum of two harmonic oscillations of different natural frequencies. **To save time, don't convert to phase-amplitude form**.

Answer: $x(t) = (15\sin t - 5\sin 3t)/4$

(1f) [30%] Determine the practical resonance frequency ω for the current equation $I'' + 2I' + 10I = 100\omega\cos(\omega t)$. [original had a typo]

Answer: $\omega = 1/\sqrt{LC} = 1/\sqrt{1/10} = \sqrt{10}$.

(1g) [30%] Determine the practical resonance frequency ω for the spring-mass system equation $2x'' + 4x'' + 10x = 100\cos(\omega t)$.

Answer: $\omega = \sqrt{k/m - c^2/(2m^2)} = \sqrt{5 - 16/8} = \sqrt{3}$.

---

2. **(ch5)** Complete enough parts to make 100%.

(2a) [40%] A non-homogeneous linear differential equation with constant coefficients has right side $f(x) = x^2e^{-x} + x^3(x+4) + e^x\cos 2x$ and characteristic equation of order 8 with roots $0, 0, 0,\ 1,\ -1, -1,\ 2i, -2i$, listed according to multiplicity. Determine the undetermined coefficients **shortest** trial solution for y_p. To save time, **do not** evaluate the undetermined coefficients and **do not** find $y_p(x)$! Undocumented detail or guessing earns no credit.

Answer: The trial solution is a linear combination of the 9 atoms $x^2e^{-x}, x^3e^{-x}, x^4e^{-x}$, $\quad x^3, x^4, x^5, x^6, x^7$, $e^x\cos x, e^x\sin x$.

(2b) [30%] Find the corrected trial solution in the method of undetermined coefficients for the differential equation $y'' + y = 3\cos x$. To save time, **do not** evaluate the undetermined coefficients and **do not** find $y_p(x)$!

Answer: The trial solution is a linear combination of the atoms $x\cos x, x\sin x$.

(2c) [50%] Assume $f(x)$ is a solution of a constant-coefficient linear homogeneous differential equation whose factored characteristic equation is $r^2(r+1)(r^2+9) = 0$. Find the shortest trial solution in the method of undetermined coefficients for the differential equation $y''' - y' = f(x)$. To save time, **do not** evaluate the undetermined coefficients and **do not** find $y_p(x)$!

Answer: $f(x)$ must be a linear combination of atoms $1, x, e^{-x}, \cos 3x, \sin 3x$. Equation $y''' - y' = 0$ has characteristic equation $r^3 - r = 0$ with roots $0, 1, -1$. The trial solution is a linear combination of the atoms x, x^2, $\quad xe^{-x}$, $\quad \cos 3x, \sin 3x$.

(2d) [50%] Find the shortest trial solution in the method of undetermined coefficients for the differential equation $y''' - y' = x + e^x$. To save time, **do not** evaluate the undetermined coefficients and **do not** find $y_p(x)$!

Answer: Characteristic equation $r^3 - r = 0$ has roots $0, 1, -1$. The trial solution is a linear combination of the atoms x, x^2, xe^x.

(2e) [25%] The general solution of a linear homogeneous differential equation with constant coefficients is

$$y = c_1e^x\cos 2x + c_2e^x\sin 2x + c_3 + c_4x + c_5x^2.$$

Find the factored form of the characteristic polynomial.

Answer: Atom $e^x\cos 2x$ is constructed from complex root $1+2i$. Because the conjugate is also a root, then $r-1-2i$ and $r-1+2i$ are factors and finally the characteristic polynomial is $a((r-1)^2+4)$ for some nonzero constant a.

(2f) [25%] Find six independent solutions of the homogeneous linear constant coefficient differential equation whose sixth order characteristic equation has roots $-1,\ -1,\ 0,\ 0,\ 2+i,\ 2-i$.

Answer: Choose 6 atoms: $e^{-x}, xe^{-x}, 1, x, e^{2x}\cos x, e^{2x}\sin x$. Cite a theorem: *Distinct atoms are independent*.

(2g) [25%] Let $f(x) = x^3 + x\sin 2x$. Find a constant-coefficient linear homogeneous differential equation which has $f(x)$ as a solution.

Answer: The atom x^3 is constructed from roots $0,0,0,0$ corresponding to factor r^4. The atom $x\sin 2x$ is constructed from roots $\pm 2i, \pm 2i$ corresponding to factor $(r^2+4)^2$. The characteristic polynomial $r^4(r^2+4)^2 = r^8 + 8r^6 + 16r^4$ belongs to DE $y^{(8)} + 8y^{(6)} + 16y^{(4)} = 0$.

(2h) [20%] Let $f(x) = 4e^x - \cosh x + e^x\cos^2 2x$. Find the characteristic polynomial of a constant-coefficient linear homogeneous differential equation which has $f(x)$ as a solution.

Answer: Because $\cosh x = (e^x + e^{-x})/2$, then roots $1, -1$ are used to produce atoms for the first two terms $4e^x$ and $-\cosh x$. Because $\cos^2 2x = (1+\cos 4x)/2$ [identity $\cos 2\theta = 2\cos^2\theta - 1$], then roots $1 \pm 4i$ and 0 are used to produce the third term $e^x\cos^2 2x$. Total roots: $1, -1, 0, 1\pm 4i$ with product of the factors $(r-1)(r+1)(r-0)((r-1)^2+16) = r^5 - 2r^4 + 16r^3 + 2r^2 - 17r$. The DE: $y^{(5)} - 2y^{(4)} + 16y''' + 2y'' - 17y' = 0$.

---

3. **(ch10)** Complete enough parts to make 100%. It is assumed that you have memorized the basic 4-item Laplace integral table and know the 6 basic rules for Laplace integrals. No other tables or theory are required to solve the problems below. If you don't know a table entry, then leave the expression unevaluated for partial credit.

(3a) [50%] Apply Laplace's method to solve the system for $x(t)$. Don't waste time solving for $y(t)$!

$$\begin{aligned} &x' = 3y, \\ &y' = 2x - y, \\ &x(0) = 0, \quad y(0) = 1. \end{aligned}$$

Answer: $x(t) = -\frac{3}{5}e^{-3t} + \frac{3}{5}e^{2t}$, $y(t) = \frac{3}{5}e^{-3t} + \frac{2}{5}e^{2t}$. Obtained by the Laplace resolvent method to save time: $(A - sI)\mathcal{L}(\mathbf{u}) = \mathbf{u}(0)$, $\mathbf{u} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$.

(3b) [25%] Find $f(t)$ by partial fraction methods, given

$$\mathcal{L}(f(t)) = \frac{3s+4}{s(s+4)}.$$

Answer: $\mathcal{L}(f(t)) = 1/s + 2/(s+4) = \mathcal{L}(1 + 2e^{-4t})$

(3c) [25%] Solve for $f(t)$, given

$$\frac{d}{ds}\mathcal{L}(f(t)) = \mathcal{L}\left(te^{2t}\right)\Big|_{s\to(s+1)}$$

Answer: $-tf(t) = e^{-t}te^{2t}$ or $f(t) = -e^t$.

(3d) [25%] Solve for $f(t)$, given

$$\mathcal{L}(f(t)) = \left(\frac{s+2}{s+1}\right)^2 \frac{1}{s+1}$$

Answer: $\mathcal{L}(f(t)) = \left(\frac{s+1}{s}\right)^2 \frac{1}{s}\Big|_{s\to(s+1)} = \frac{1}{s} + \frac{2}{s^2} + \frac{1}{s^3}\Big|_{s\to(s+1)} = \mathcal{L}((1+2t+t^2/2)e^{-t})$.

---

4. **(ch10)** Complete enough parts to make 100%.

(4a) [25%] Fill in the blank spaces in the Laplace table:

$f(t)$	t^3			$e^t \cos t$	t^2e^{-t}
$\mathcal{L}(f(t))$	$\frac{6}{s^4}$	$\frac{1}{2s+2}$	$\frac{s+1}{s^2+2s+10}$		

Answers: Left to right: $\frac{1}{2}e^{-t}$, $e^{-t}\cos 3t$, $\frac{s-1}{(s-1)^2+1}$, $\frac{2}{(s+1)^3}$.

(4b) [50%] Solve by Laplace's method for the solution $x(t)$:

$$x''(t) + x'(t) = 9e^{-t}, \quad x(0) = x'(0) = 0.$$

Answer: $x(t) = -(9+9t)e^{-t} + 9$.

(4c) [25%] Solve for $x(t)$, given

$$\mathcal{L}(x(t)) = \frac{d}{ds}\left(\mathcal{L}(e^{2t}\sin 2t)\right) + \mathcal{L}(t\sin t)|_{s\to(s+2)}.$$

Answer: $x(t) = -te^{2t}\sin 2t + te^{-2t}\sin t$.

(4d) [25%] Solve for $x(t)$, given

$$\mathcal{L}(x(t)) = \frac{s+2}{(s+1)^2} + \frac{1+s}{s^2+5s}$$

Answer: $\mathcal{L}(x(t)) = \frac{s+1}{s^2}\Big|_{s\to(s+1)} + \frac{1}{5s} + \frac{4}{5(s+4)} = \mathcal{L}\left((1+t)e^{-t} + \frac{1}{5} + \frac{4}{5}e^{-4t}\right)$.

(4e) [25%] Find $\mathcal{L}(f(t))$, given $f(t) = e^{2t}\left(\frac{\sin(t)}{t}\right)$.

Answer: Let $g(t) = (1/t)\sin t$. Then $\frac{d}{ds}\mathcal{L}(g(t)) = \mathcal{L}(-tg(t)) = \mathcal{L}(-\sin t) = -1/(s^2+1)$. Solve $\frac{dG(s)}{ds} = \frac{-1}{s^2+1}$ to get $G(s) = \frac{\pi}{2} - \arctan s$. Then $L(f(t)) = \mathcal{L}(g(t))|_{s\to(s-2)} = \frac{\pi}{2} - \arctan(s-2)$.

(4f) [30%] Apply Laplace's method to find a formula for $\mathcal{L}(x(t))$. **Do not** solve for $x(t)$! Document steps by reference to tables and rules.

$$\frac{d^4x}{dt^4} + 4\frac{d^2x}{dt^2} = e^t(5t + 4e^t + 3\sin 3t), \quad x(0) = x'(0) = x''(0) = 0, \quad x'''(0) = -1.$$

Answer: $\mathcal{L}(x(t)) = p/q$, $p = 1 + \mathcal{L}(e^t(5t + 4e^t + 3\sin 3t))$, $q = s^4 + 4s^2$. Expanding, $p = 1 + \left(\frac{5}{s^2} + \frac{4}{s-1} + \frac{9}{s^2+9}\right)\Big|_{s\to(s-1)} = 1 + \frac{5}{(s-1)^2} + \frac{4}{s-2} + \frac{9}{(s-1)^2+9}$.

(4g) [25%] Find $\mathcal{L}(f(t))$, given $f(t) = u(t-\pi)\dfrac{\sin(t)}{t-\pi}$, where u is the unit step function [original had a typo].

Answer: Use the second shifting theorem $\mathcal{L}(u(t-a)g(t)) = e^{-as}\mathcal{L}(g(t+a))$. Then $\mathcal{L}(f(t)) = e^{-\pi s}\mathcal{L}\left(\frac{\sin(t+\pi)}{t-\pi+\pi}\right) = e^{-\pi s}\mathcal{L}\left(\frac{-\sin t}{t}\right)$. Let $h(t) = \frac{-\sin t}{t}$, then $\mathcal{L}(th(t)) = \frac{-1}{s^2+1}$ implies a first order DE $-\frac{dH}{ds} = \frac{-1}{s^2+1}$, which can be solved to get $H(s) = \arctan(s) - \frac{\pi}{2}$. Then $\mathcal{L}(f(t)) = e^{-\pi s}(\arctan(s) - \pi/2)$.

(4h) [40%] Find $f(t)$ by partial fraction methods, given

$$\mathcal{L}(f(t)) = \frac{8s^2 - 24}{(s-1)(s+3)(s+1)^2}.$$

Answer: $\mathcal{L}(f(t)) = \frac{-1}{s-1} + \frac{-3}{s+3} + \frac{4}{(s+1)^2} + \frac{4}{s+1} = \mathcal{L}\left(-e^t - 3e^{-3t} + 4te^{-t} + 4e^{-t}\right)$.

(4i) [30%] Apply Laplace's method to find a formula for $\mathcal{L}(x(t))$. To save time, **do not** solve for $x(t)$! Document steps by reference to tables and rules.

$$x^{(4)} - x^{(2)} = 3t^2 + 4e^{-2t} + 5e^t\sin 2t, \quad x(0) = x'(0) = x''(0) = 0, \quad x'''(0) = -1.$$

Answer: $\mathcal{L}(x(t)) = p/q$, $p = -1 + \mathcal{L}(\text{RHS})$, $q = s^4 - s^2$. Finally $p = -1 + \mathcal{L}(3t^2 + 4e^{-2t} + 5e^t\sin 2t) = -1 + \frac{6}{s^3} + \frac{4}{s+2} + \frac{10}{(s-1)^2+4}$.

---

5. **(ch6)** Complete all of the items below.

(5a) [30%] Find the eigenvalues of the matrix $A = \begin{pmatrix} -2 & 7 & 1 & 12 \\ -1 & 6 & -3 & 15 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & -2 & 3 \end{pmatrix}$. To save time, **do not** find eigenvectors!

Answer: $-1, 5, 3 \pm 2i$

(5b) [40%] Given $A = \begin{pmatrix} 5 & 1 & -1 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix}$, assume there exists an invertible matrix P and a diagonal matrix D such that $AP = PD$. Circle all possible columns of P from the list below.

$$\begin{pmatrix} 2 \\ -3 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

Answer: Matrix P must contain eigenvectors of P corresponding to eigenvalues 2, 4, 5. For each given vector $\mathbf{v}$, multiply $A\mathbf{v}$ and see if it is $\lambda\mathbf{v}$ for some λ. The first works fir $\lambda = 2$ and the others do not work.

(5c) [30%] Find all eigenpairs for the matrix $A = \begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix}$. Then display Fourier's model for A.

Answer: $\left(0, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right), \left(3, \begin{pmatrix} 1 \\ -2 \end{pmatrix}\right)$.

(5d) [50%] Find the remaining eigenpairs of

$$E = \begin{pmatrix} 6 & 2 & -2 \\ 0 & 5 & 1 \\ 0 & 1 & 5 \end{pmatrix}$$

provided we already know one eigenpair

$$\left(6, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right).$$

Answer: Eigenvalues are 4, 6, 6 with corresponding eigenvectors $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

(5e) [25%] Suppose a 2×2 matrix A has eigenpairs $\left(3, \begin{pmatrix} 3 \\ 1 \end{pmatrix}\right), \left(-2, \begin{pmatrix} 1 \\ -2 \end{pmatrix}\right)$. Display an invertible matrix P and a diagonal matrix D such that $AP = PD$.

Answer: Define $P = \begin{pmatrix} 3 & 1 \\ 1 & -2 \end{pmatrix}$, $D = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}$. Then $AP = PD$.

(5f) [25%] Assume the vector general solution $\mathbf{x}(t)$ of the linear differential system $\mathbf{x}' = A\mathbf{x}$ is given by

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} + c_3 e^{2t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Display Fourier's model for the 3×3 matrix A.

Answer: $A \left(c_1 \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) = 0c_1 \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} + 2c_2 \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} + 2c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$

(5g) [30%] Find the eigenvalues of the matrix $A = \begin{pmatrix} -2 & 7 & 1 & 27 \\ -1 & 6 & -3 & 62 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & -1 & 0 \end{pmatrix}$. To save time, **do not** find eigenvectors!

Answer: Expand by cofactors along column 1. The eigenvalues are $-1, 1, 2, 5$.

(5h) [30%] Assume A is 2×2 and Fourier's model holds:

$$A\left(c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right) = 2c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Find A.

Answer: $AP = PD$ implies $A = PDP^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} .5 & .5 \\ .5 & -.5 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$.

(5i) [40%] Let $A = \begin{pmatrix} 3 & 0 & -1 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$. Circle the possible eigenvectors of A in the list below.

$$\begin{pmatrix} -4 \\ 2 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Answer: Fourier's model does not hold [A is not diagonalizable] because there are only two eigenvectors $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ for eigenvalue $\lambda = 3$. The first is a linear combination of these eigenvectors, hence itself an eigenvector. The second is one already reported, The third is not an eigenvector. The problem should be solved by testing the equation $A\mathbf{v} = 3\mathbf{v}$ for each of the 3 vectors v in the list, not by doing the eigenanalysis of A.

(5j) [40%] Consider the 3×3 matrix

$$E = \begin{pmatrix} 4 & 2 & -2 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix}.$$

Show that matrix E has a Fourier model: [original had a typo]

$$E\left(c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}\right) = 4c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 4c_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + 2c_3 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}.$$

Answer: Do the eigenanalysis of A. Alternate: verify that the eigenpairs extracted from Fourier's model actually work, which involves 3 matrix multiplies.

(5k) [40%] Define E as in the previous problem. Find $E^3 \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$ without using matrix multiply.

Answer: Vector $\mathbf{v} \equiv \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} = \mathbf{v}_2 + \mathbf{v}_3$, where $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are the eigenvectors in Fourier's model. Then $E^3\mathbf{v} = 4^3\mathbf{v}_2 + 2^3\mathbf{v}_3 = \begin{pmatrix} 16 \\ 56 \\ 72 \end{pmatrix}$.

(5l) [30%] Let $P = \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix}$, $D = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}$ and define A by $AP = PD$. Display the eigenpairs of A.

Answer: $\left(3, \begin{pmatrix} 3 \\ 1 \end{pmatrix}\right)$, $\left(-2, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right)$

(5m) [30%] Assume the vector general solution $\mathbf{x}(t)$ of the linear differential system $\mathbf{x}' = M\mathbf{x}$ is given by

$$\mathbf{x}(t) = c_1 e^{-t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Display an invertible matrix P and a diagonal matrix D such that $MP = PD$.

Answer: The eigenvalues come from the coefficients in the exponentials, -1 and 2. The eigenpairs are $\left(-1, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right)$, $\left(2, \begin{pmatrix} -1 \\ 1 \end{pmatrix}\right)$. Then $P = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$.