Name KEY

Draft

Applied Differential Equations 2250

Exam date: Thursday, 26 March, 2009

Instructions: This in-class exam is 50 minutes. Up to 30 extra minutes will be given. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%.

1. **(Frame sequences and the 3 possibilities)**
Determine a, b such that the system has a unique solution, infinitely many solutions, or no solution:

$$\begin{array}{rcrcrcl} x & + & 6y & + & 2z & = & 2b \\ 4x & + & 24y & + & 6z & = & 2+b \\ 3x & + & 3ay & + & 4z & = & b \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 6 & 2 & 2b \\ 4 & 24 & 6 & 2+b \\ 3 & 3a & 4 & b \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 6 & 2 & 2b \\ 0 & 0 & -2 & 2-7b \\ 3 & 3a & 4 & b \end{array}\right) \quad \text{combo}(1,2,-4)$$

$$\left(\begin{array}{ccc|c} 1 & 6 & 2 & 2b \\ 0 & 0 & -2 & 2-7b \\ 0 & 3a-18 & -2 & -5b \end{array}\right) \quad \text{combo}(1,3,-3)$$

$$\left(\begin{array}{ccc|c} 1 & 6 & 2 & 2b \\ 0 & 0 & -2 & 2-7b \\ 0 & 3a-18 & 0 & -2+2b \end{array}\right) \quad \text{combo}(2,3,-1)$$

$$\left(\begin{array}{ccc|c} 1 & 6 & 2 & 2b \\ 0 & 3a-18 & 0 & -2+2b \\ 0 & 0 & -2 & 2-7b \end{array}\right) \quad \text{swap}(2,3)$$

always lead vars x, z. Free var y provided $3a-18=0$.

ans: unique solution for $3a-18 \neq 0$, all b.
No solution if $3a-18=0$ and $-2+2b \neq 0$.
∞-many solutions if $3a-18=0$ and $-2+2b=0$

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2. **(vector spaces)** Do all parts.

(a) [50%] Let V be the vector space of all column vectors $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and let S be the subset of V given by the equations $x_1(x_2+x_3)=0$, $x_2+2x_3=0$. Prove or disprove that S is a subspace of V.

(b) [50%] Find a basis of 4-vectors for the subspace of $\mathcal{R}^4$ given by the system of restriction equations

$$\begin{array}{rcrcrcrcl} x_1 & + & 6x_2 & - & x_3 & - & 4x_4 & = & 0, \\ x_1 & + & 4x_2 & - & 2x_3 & - & 4x_4 & = & 0, \\ & & 4x_2 & + & 2x_3 & & & = & 0 \end{array}$$

(a) Apply not a Subspace Theorem part (2)

(1) $\vec{0}$ not in S or (2) $\vec{x}$ and $\vec{y}$ in S but $\vec{x}+\vec{y}$ not in S or (3) $\vec{x}$ in S but $-\vec{x}$ not in S.

Let $\vec{x}=\begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$, $\vec{y}=\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. Both are in S. But $\vec{x}+\vec{y}=\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ is not in S.

(b) $A=\left(\begin{array}{cccc|c} 1 & 6 & -1 & -4 & 0 \\ 1 & 4 & -2 & -4 & 0 \\ 0 & 4 & 2 & 0 & 0 \end{array}\right)$ $\quad \text{rref}(A)=\left(\begin{array}{cccc|c} 1 & 0 & -4 & -4 & 0 \\ 0 & 1 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$

$$\begin{cases} x_1 = 4t_1+4t_2 \\ x_2 = -t_1/2 \\ x_3 = t_1 \\ x_4 = t_2 \end{cases}$$ scalar sol

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = t_1\begin{pmatrix} 4 \\ -1/2 \\ 1 \\ 0 \end{pmatrix} + t_2\begin{pmatrix} 4 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$ vector sol

Basis $=\left\{ \begin{pmatrix} 4 \\ -1/2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

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3. **(independence)** Do **all parts.**

(a) [50%] Let $\mathbf{u}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 2 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 4 \end{pmatrix}$, $\mathbf{u}_3 = \begin{pmatrix} 2 \\ -1 \\ 3 \\ 6 \end{pmatrix}$, State a test that can decide independence or dependence of this list of three vectors [20%]. Apply the test and report the result [30%].

(b) [50%] Extract from the list below a largest set of independent vectors.

$$\mathbf{a} = \begin{pmatrix} 2 \\ 0 \\ -2 \\ 2 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}. \mathbf{c} = \begin{pmatrix} 3 \\ 0 \\ -1 \\ 5 \end{pmatrix}, \mathbf{d} = \begin{pmatrix} 3 \\ 0 \\ 3 \\ 9 \end{pmatrix}, \mathbf{e} = \begin{pmatrix} 0 \\ 0 \\ 2 \\ 2 \end{pmatrix},$$

(a) $\boxed{\text{Test}}$: Let $A = \text{aug}(\vec{u}_1, \vec{u}_2, \vec{u}_3)$. Then $\vec{u}_1, \vec{u}_2, \vec{u}_3$ are independent $\Leftrightarrow$ rank$(A) = 3$.

$$A = \begin{pmatrix} 1 & 1 & 2 \\ -1 & 0 & -1 \\ 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix} \to A_1 = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix} \to A_2 = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix} \to A_3 = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\to A_4 = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$ Then rank$(A) = 2 \Rightarrow \boxed{u_1, u_2, u_3 \text{ dependent}}$

(b) $$A = \begin{pmatrix} 2 & 1 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -2 & -1 & -1 & 3 & 2 \\ 2 & 1 & 5 & 9 & 2 \end{pmatrix} \quad \text{rref}(A) = \begin{pmatrix} 1 & 1/2 & 0 & -3 & -3/2 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

pivots = 1, 3 $\boxed{\vec{a}, \vec{c}}$

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4. (determinants) Do all parts.

(a) [40%] Assume given 3×3 matrices A, B. Suppose $E_5E_4B = E_3E_2E_1A$ and E_1, E_2, E_3, E_4, E_5 are elementary matrices representing respectively a swap, a multiply by 2, a swap, a combination, and a multiply by 6. Assume $\det(A) = 2$. Find $\det(3AB^2)$.

(b) [20%] Determine all values of x for which B^{-1} fails to exist, where B is the transpose of A and matrix A is given by $A = \begin{pmatrix} 1 & 2x+1 & 3x-5 \\ 2 & 0 & 3 \\ 5x & 0 & 10 \end{pmatrix}$.

(c) [40%] Apply the adjugate formula for the inverse to find the value of the entry in row 1, column 3 of A^{-1}, given A below. Other methods are not acceptable.

$$A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ -1 & 0 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 2 \end{pmatrix}$$

(a) $|E_5||E_4||B| = |E_3||E_2||E_1||A|$ by det. prod. Thm

$(6)(1)|B| = (-1)(2)(-1)|A|$ because $|\text{swap}| = -1$, $|\text{combo}| = 1$ and $|\text{mult by } m| = m$

$|B| = \frac{1}{3}|A| = \frac{2}{3}$

$\det(3AB^2) = |3I||A||B|^2 = (27)(2)\left(\frac{2}{3}\right)^2 = (3)(2)(4) = \boxed{24}$

(b) B^{-1} fails to exist $\iff \det(A) = 0 \iff -(2x+1)(20-15x) = 0 \iff \boxed{x = \frac{-1}{2} \text{ or } x = \frac{4}{3}}$

(c) Entry 1, 3 of $A^{-1} = \dfrac{\text{Cofactor}(A, 3, 1)}{|A|} = \dfrac{(-1)^4 \begin{vmatrix} 2 & 0 & 1 \\ 0 & -1 & 1 \\ 2 & 0 & 2 \end{vmatrix}}{|A|} = \dfrac{1(-2)}{1} = \boxed{-2}$

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5. **(Linear differential equations)** Sample for 2 April: Do all three parts.
(a) [30%] Solve for the general solution of $y'' + 2y' + y = 0$.
(b) [40%] The characteristic equation is $r^3(r-2)^2(r^2+2r+5) = 0$. Find the general solution y of the homogeneous constant-coefficient differential equation.
(c) [30%] A second order differential equation $y'' + py' + qy = 0$ with constant coefficients p, q has two solutions e^x and e^{-2x}. Find p and q.

(a) $r^2 + 2r + 1 = 0$

$(r+1)^2 = 0$

$r = -1, -1$

atoms $= e^{-x}, xe^{-x}$ by Euler's Thm

$$\boxed{y = c_1 e^{-x} + c_2 x e^{-x}}$$

(b) roots $= 0, 0, 0, \quad 2, 2, \quad -1+2i, -1-2i$

Base atoms $= e^{0x} \quad e^{2x} \quad e^{-x}\cos 2x, \; e^{-x}\sin 2x$

atoms $= 1, x, x^2, \quad e^{2x}, xe^{2x}, \quad e^{-x}\cos 2x, \; e^{-x}\sin 2x$ used Euler's Thm

$y =$ linear combination of the above 7 atoms

(c) e^x, e^{-2x} = atoms

$1, -2$ = roots of char. eq., by Euler's Thm

$(r-1), (r+2)$ = Factors by root-factor Thm of College algebra

$(r-1)(r+2) = 0$ char eq

$r^2 + r - 2 = 0$

$$\boxed{y'' + y' - 2y = 0} \quad \boxed{\begin{matrix} p = 1 \\ q = -2 \end{matrix}}$$

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