Name KEY

2250 Midterm 2 [7:30, S2009]

Applied Differential Equations 2250

Draft original

Exam date: Thursday, 26 March, 2009

Instructions: This in-class exam is 50 minutes. Up to 30 extra minutes will be given. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%.

1. (Frame sequences and the 3 possibilities)
Determine a, b such that the system has a unique solution, infinitely many solutions, or no solution:

$$\begin{aligned} x &+ 2y &+ 2z &= -2b \\ 3x &+ ay &+ 4z &= -b \\ 4x &+ 8y &+ 6z &= 2-b \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 2 & -2b \\ 3 & a & 4 & -b \\ 4 & 8 & 6 & 2-b \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 2 & -2b \\ 0 & a-6 & -2 & 5b \\ 4 & 8 & 6 & 2-b \end{array}\right) \quad \text{combo}(1,2,-3)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 2 & -2b \\ 0 & a-6 & -2 & 5b \\ 0 & 0 & -2 & 2+7b \end{array}\right) \quad \text{combo}(1,3,-4)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 2 & -2b \\ 0 & a-6 & 0 & -2-2b \\ 0 & 0 & -2 & 2+7b \end{array}\right) \quad \text{combo}(3,2,-1)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 2+5b \\ 0 & a-6 & 0 & -2-2b \\ 0 & 0 & -2 & 2+7b \end{array}\right) \quad \text{combo}(3,1,1)$$

always 2 lead vars x, z

Free var if $a-6=0$. Consistent if $a-6=0=-2-2b$.

No sol if $a-6=0$, $0 \neq -2-2b$

ans: unique sol $\iff$ $a-6\neq 0$, b arbitrary

No sol $\iff$ $a-6=0$, $-2-2b\neq 0$

∞-many sols $\iff$ $a-6=0$, $-2-2b=0$

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2. **(vector spaces)** Do all parts.

(a) [50%] Let V be the vector space of all column vectors $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and let S be the subset of V given by the equations $x_1 x_3 = 0$, $x_2 + 2x_3 = 0$. Prove or disprove that S is a subspace of V.

(b) [50%] Find a basis of 4-vectors for the subspace of $\mathcal{R}^4$ given by the system of restriction equations

$$\begin{array}{rcrcrcrcl} x_1 & + & 6x_2 & - & x_3 & + & 2x_4 & = & 0, \\ x_1 & + & 4x_2 & - & 2x_3 & + & 2x_4 & = & 0, \\ & & 4x_2 & + & 2x_3 & & & = & 0 \end{array}$$

(a) Not a subspace Thm: (2) $\vec{x}$ in S, $\vec{y}$ in S but $\vec{x}+\vec{y}$ not in S

Let $\vec{x} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$, $\vec{y} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ Then $\vec{x}$ and $\vec{y}$ in S but $\vec{x}+\vec{y} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ is not.

(b) $C = \left(\begin{array}{cccc|c} 1 & 6 & -1 & 2 & 0 \\ 1 & 4 & -2 & 2 & 0 \\ 0 & 4 & 2 & 0 & 0 \end{array}\right)$

$\mathrm{rref}(C) = \left(\begin{array}{cccc|c} 1 & 0 & -4 & 2 & 0 \\ 0 & 1 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$

$\begin{cases} x_1 = 4t_1 - 2t_2 \\ x_2 = -\frac{1}{2}t_1 \\ x_3 = t_1 \\ x_4 = t_2 \end{cases}$ scalar sol

$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = t_1 \begin{pmatrix} 4 \\ -1/2 \\ 1 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

vector sol

Basis $= \left\{ \begin{pmatrix} 4 \\ -1/2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

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3. (independence) Do **all parts.**

(a) [50%] Let $\mathbf{u}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 2 \end{pmatrix}$, $\mathbf{u}_3 = \begin{pmatrix} 2 \\ -1 \\ 3 \\ 3 \end{pmatrix}$, State a test that can decide independence or dependence of this list of three vectors [20%]. Apply the test and report the result [30%].

(b) [50%] Extract from the list below a largest set of independent vectors.

$\mathbf{a} = \begin{pmatrix} 2 \\ 0 \\ -2 \\ 2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$. $\mathbf{c} = \begin{pmatrix} 3 \\ 0 \\ 3 \\ 9 \end{pmatrix}$, $\mathbf{d} = \begin{pmatrix} 0 \\ 0 \\ 2 \\ 2 \end{pmatrix}$, $\mathbf{e} = \begin{pmatrix} 3 \\ 0 \\ -1 \\ 5 \end{pmatrix}$,

(a) Test : $\vec{u}_1, \vec{u}_2, \vec{u}_3$ independent $\Leftrightarrow$ $A = \text{aug}(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ has rank $= 3$

$$A = \begin{pmatrix} 1 & 1 & 2 \\ -1 & 0 & -1 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \to A_1 = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \xrightarrow{\text{combo}} A_2 = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix} \text{combo}$$

$$\to A_3 = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \text{combo} \to A_4 = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{combo.}$$

$\boxed{\text{Rank} = 2 \Rightarrow \text{Dependent}}$

(b) $$A = \begin{pmatrix} 2 & 1 & 3 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ -2 & -1 & 3 & 2 & -1 \\ 2 & 1 & 9 & 2 & 5 \end{pmatrix} \quad \text{rref}(A) = \begin{pmatrix} 1 & 1/2 & 0 & -1/2 & 1 \\ 0 & 0 & 1 & 1/3 & 1/3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

pivots $= 1, 3$ $\quad\boxed{\vec{a}, \vec{c}}$

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4. **(determinants)** Do all parts.

(a) [40%] Assume given 3×3 matrices A, B. Suppose $E_5E_4B = E_3E_2E_1A$ and E_1, E_2, E_3, E_4, E_5 are elementary matrices representing respectively a swap, a multiply by 2, a swap, a combination, and a multiply by 5. Assume $\det(A) = 3$. Find $\det(2AB^2)$.

(b) [20%] Determine all values of x for which B^{-1} fails to exist, where B is the transpose of A and matrix A is given by $A = \begin{pmatrix} 1 & 2x-1 & 3x-5 \\ 2 & 0 & 2 \\ 5x & 0 & 10 \end{pmatrix}$.

(c) [40%] Apply the adjugate formula for the inverse to find the value of the entry in row 4, column 3 of A^{-1}, given A below. Other methods are not acceptable.

$$A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ -1 & 0 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 2 \end{pmatrix}$$

(a) $|E_5||E_4||B| = |E_3||E_2||E_1||A|$ by det. prod. rule

$(5)\ (1)\ |B| = (-1)\ (2)\ (-1)\ |A|$

Because $\begin{cases} |\text{swap}| = -1 \\ |\text{combo}| = 1 \\ |\text{mult}| = \text{multiplier} \end{cases}$

$|B| = \frac{6}{5}$ Because $|A|=3$

$|2AB^2| = |2I\,A\,B\,B|$

$= |2I|\,|A|\,|B|\,|B|$

$= \boxed{(8)(3)\left(\frac{6}{5}\right)^2}$

(b) B^{-1} fails to exist $\iff$ $\det(A)=0$ $\iff$ $-(2x-1)(20-10x)=0$ $\iff$ $\boxed{\begin{cases} x=\frac{1}{2} \\ \text{or} \\ x=2 \end{cases}}$

(c) entry 4,3 of A^{-1} $= \dfrac{\text{entry } 4,3 \text{ of } \operatorname{adj}(A)}{|A|} = \dfrac{\text{Cofactor}(A,3,4)}{|A|}$

$= \dfrac{(-1)^{3+4}\ \text{minor}(A,3,4)}{|A|} = \boxed{0}$

$\text{minor} = \begin{vmatrix} 1 & 2 & 0 \\ -1 & 0 & 1 \\ 1 & 2 & 0 \end{vmatrix} = (1)(-1)\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} = 0$

Don't need to find det(A)! It's value is $|A|=1$.

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5. **(Linear differential equations)** Sample for 2 April: Do all three parts.
(a) [30%] Solve for the general solution of $y'' + 2y' + y = 0$.
(b) [40%] The characteristic equation is $r^3(r-2)^2(r^2+2r+5) = 0$. Find the general solution y of the homogeneous constant-coefficient differential equation.
(c) [30%] A second order differential equation $y'' + py' + qy = 0$ with constant coefficients p, q has two solutions e^x and e^{-2x}. Find p and q.

(a) $r^2+2r+1=0$

$(r+1)^2=0$

$r=-1,-1$

atoms $= e^{-x}, xe^{-x}$ by Euler's Thm

$\boxed{y = c_1e^{-x} + c_2xe^{-x}}$

(b) roots $= 0,0,0, \quad 2,2, \quad -1+2i, -1-2i$

Base atoms $= e^{0x} \quad e^{2x} \quad e^{-x}\cos 2x, e^{-x}\sin 2x$

atoms $= 1, x, x^2, \quad e^{2x}, xe^{2x}, \quad e^{-x}\cos 2x, e^{-x}\sin 2x$ used Euler's Thm

$y =$ linear combination of the above 7 atoms

(c) e^x, e^{-2x} = atoms

$1, -2$ = roots of char. eq., by Euler's Thm

$(r-1), (r+2)$ = Factors by root-factor Thm of College algebra

$(r-1)(r+2) = 0$ char eq

$r^2+r-2=0$

$\boxed{y''+y'-2y=0}$ $\boxed{\begin{matrix}p=1\\ q=-2\end{matrix}}$

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