Name KEY

Differential Equations and Linear Algebra 2250

Midterm Exam 1 [10:45]
Thursday, 19 February 2009

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

1. **(Quadrature Equations)**

   (a) [25%] Solve $y' = \dfrac{x+2x^2}{1+x}$.

   (b) [25%] Solve $y' = (\sin x + 2\cos x)(2\sin x - \cos x)$.

   (c) [25%] Solve $y' = x\tan\left(\pi + x^2\right)$, $y(0) = 2$.

   (d) [25%] Find the position $x(t)$ from the velocity model $\dfrac{d}{dt}(e^{-t}v) = 0$, $v(0) = 10$ and the position model $\dfrac{dx}{dt} = v(t)$, $x(0) = -1$.

(a)

$$\begin{array}{r|l} & 2x-1 \\ \hline x+1 & 2x^2+x \\ & \underline{2x^2+2x} \\ & -x \\ & \underline{-x-1} \\ & 1 \end{array}$$

$$y = \int \frac{x+2x^2}{1+x}\,dx = \int \left(2x-1 + \frac{1}{x+1}\right)dx$$

$$y = x^2 - x + \ln|x+1| + c$$

(b)

$$y = \int (\sin x + 2\cos x)(2\sin x - \cos x)\,dx$$

$$= \int u(-du) \qquad u = \sin x + 2\cos x,\ du = (\cos x - 2\sin x)dx$$

$$= -u^2/2 + c$$

$$= -\tfrac{1}{2}(\sin x + 2\cos x)^2 + c$$

(c)

$$y = \int x\tan(\pi + x^2)\,dx$$

$$= \int \tan(u)\,\frac{du}{2} \qquad u = \pi + x^2,\ du = 2x\,dx$$

$$= -\ln|\cos u| + c$$

$$= -\ln|\cos(\pi + x^2)| + c$$

(d)

$e^{-t}v = c$, $e^0 v(0) = c$, $c = 10$.

$$\boxed{v = 10e^t}$$

$$x' = 10e^t$$

$$\boxed{x = 10e^t - 11}$$

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2. (Classification of Equations)
The differential equation $y' = f(x,y)$ is defined to be **separable** provided $f(x,y) = F(x)G(y)$ for some functions F and G.

(a) [40%] Check (☒) the problems that can be put into separable form. No details expected.

☒	$y' + xy = y(2x^2 + e^x) + x^3y$	☐	$y' = (x+1)(y-1) - (x-1)y$ $= xy + y - x - 1 - xy + y$
☐	$y' = 3e^{2x-y}e^{3y} + 2e^{3x+3y}$	☒	$y' + e^y = e^{x+y}$

Linear
not Separable

(b) [10%] Is $y' + 2xy = ye^x$ linear? No details expected.

(c) [10%] Give an example of $y' = f(x,y)$ which is separable but not linear and not quadrature. No details expected.

(d) [40%] Apply tests to show that $y' = e^x + e^y$ is not separable, not linear and not quadrature. Supply all details. Let $f(x,y) = e^x + e^y$, below.

(b) Yes. Form is $y' + py = q$ where $p = 2x - e^x$, $q = 0$

(c) $y' = y^2$

(d) $\frac{f_x}{f} = \frac{e^x}{e^x + e^y}$ is not independent of y $\Rightarrow$ $y' = f(x,y)$ not Separable.

$\frac{\partial f}{\partial y} = e^y$ not independent of y $\Rightarrow$ not linear

$\frac{\partial f}{\partial y} \neq 0$ $\Rightarrow$ not quadrature

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3. (Solve a Separable Equation)
Given $(x+1)(y+1)y' = \Big((2x+2)e^{-x+2} + 2x^2 + 3\Big)(y-1)(y-2)$.

Find a non-equilibrium solution in implicit form.
To save time, **do not solve** for y explicitly and **do not solve** for equilibrium solutions.

$$\int \frac{(y+1)\,y'\,dx}{(y-1)(y-2)} = \int \left(2e^{2-x} + \frac{2x^2+3}{x+1}\right) dx \quad \leftarrow \text{use } \frac{y'}{G} = F \text{ with Quadrature}$$

$$\int \left(\frac{A}{y-1} + \frac{B}{y-2}\right) dy = -2e^{2-x} + \int \left(2x-2+\frac{5}{x+1}\right) dx$$

$$\begin{array}{r|l} & 2x-2 \\ \hline x+1 & 2x^2+3 \\ & 2x^2+2x \\ \hline & 3-2x \\ & -2-2x \\ \hline & 5 \end{array}$$

$$A\ln|y-1| + B\ln|y-2| = -2e^{2-x} + x^2 - 2x + \ln|x+1|^5 + C$$

$$\frac{y+1}{(y-1)(y-2)} = \frac{A}{y-1} + \frac{B}{y-2}$$

$$y+1 = A(y-2) + B(y-1)$$

Sampling

$y=1$: $2 = -A + 0$

$y=2$: $3 = 0 + B$

$$\boxed{-2\ln|y-1| + 3\ln|y-2| = -2e^{2-x} + x^2 - 2x + \ln|x+1|^5 + C}$$

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4. (Linear Equations)

(a) [60%] Solve the linear model $3x'(t) = -96 + \frac{15}{2t+3}x(t)$, $x(0) = 32$. Show all integrating factor steps.

(b) [20%] Solve the homogeneous equation $\frac{dy}{dx} - (\sin x)y = 0$.

(c) [20%] Solve $5\frac{dy}{dx} + 3y = 6$ using the superposition principle $y = y_h + y_p$. Expected are answers for y_h and y_p.

(a) $x' + \left(\frac{-5}{2t+3}\right)x = -32$

$\frac{(xe^u)'}{e^u} = -32$ Detail $\longleftrightarrow$

$u = \int \frac{-5}{2t+3}\,dt$

$u = -\frac{5}{2}\ln|2t+3|$ [drop c]

$e^u = (2t+3)^{-5/2}$ because $x(0)=32$

$(xe^u)' = -32e^u$

$xe^u = -32\int (2t+3)^{-5/2}\,dt$

$xe^u = -32\,\frac{(2t+3)^{-3/2}}{(-3/2)(2)} + c$

$x = \frac{-32}{-3}(2t+3) + c(2t+3)^{5/2}$

$32 = \frac{-32}{-3}(3) + c(3)^{5/2} \Rightarrow c = 0.$

$$\boxed{x = \frac{64}{3}t + 32}$$

(b) $y = \frac{c}{e^u}$ where $u = \int(-\sin x)\,dx = \cos x$ $\boxed{y = ce^{-\cos x}}$

(c) $y_p = 2$

$y_h = \frac{c}{e^u}$ where $u = \int \frac{3}{5}\,dx = 3x/5$

$y = y_p + y_h$

$$\boxed{y = 2 + ce^{-3x/5}}$$

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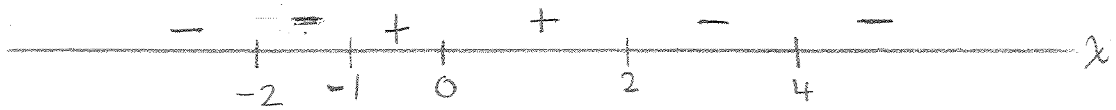
5. (Stability)

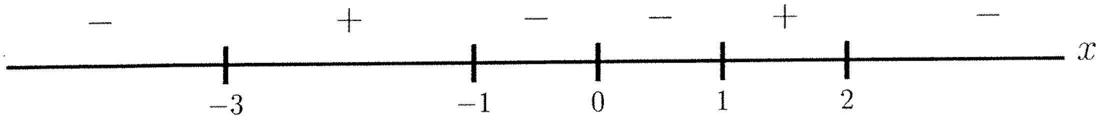
(a) [50%] Draw a phase line diagram for the differential equation

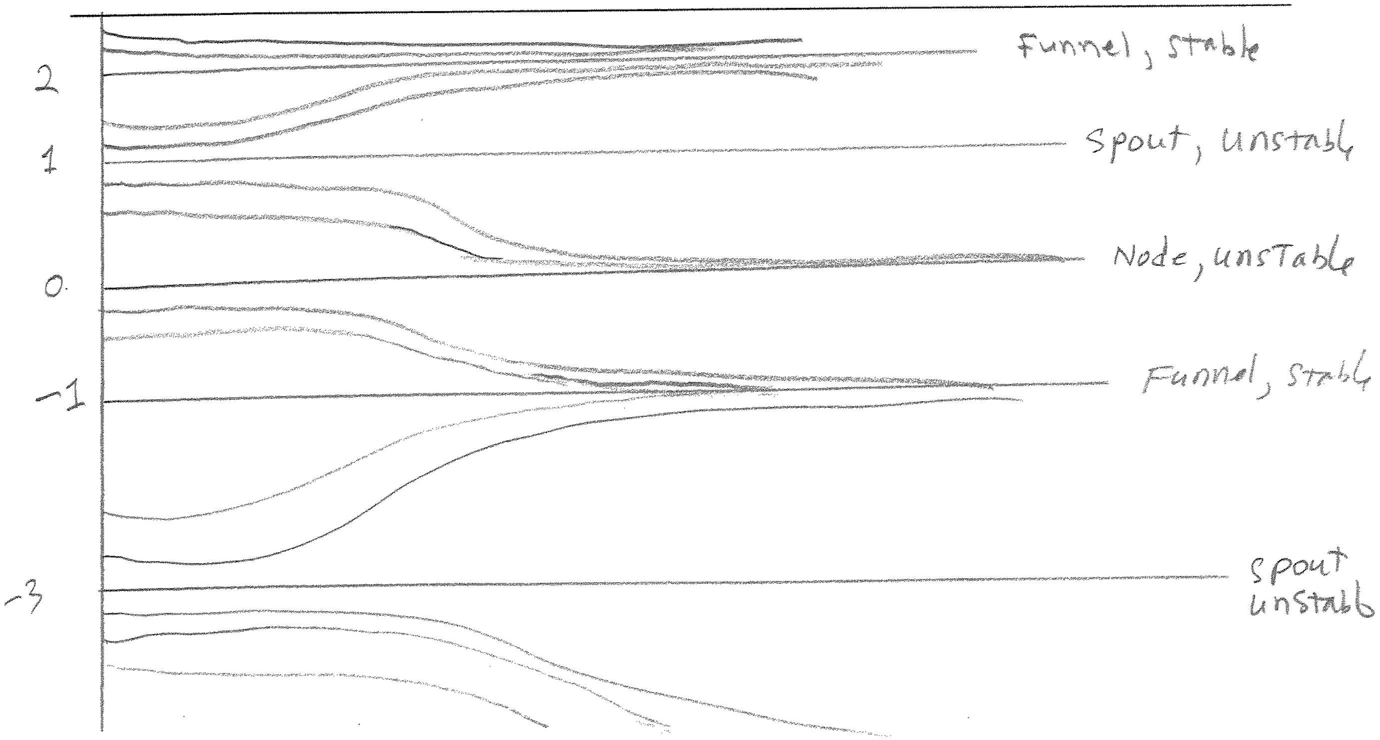
$$\frac{dx}{dt} = \ln(1+4x^2)\,(5-|2x-3|)^3\,(2+x)(4-x)(4-x^2)^3 e^{\sin x}.$$

Expected in the phase line diagram are equilibrium points and signs of dx/dt.



(b) [50%] Assume an autonomous equation $x'(t) = f(x(t))$. Draw a phase diagram with at least 12 threaded curves, using the phase line diagram given below. Add these labels as appropriate: funnel, spout, node [neither spout nor funnel], stable, unstable.





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ans: Funnel
Spout
Node
Funnel
Spout