Name KEY

Differential Equations and Linear Algebra 2250

Midterm Exam 1 [7:30]

Thursday, 19 February 2009

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

1. **(Quadrature Equations)**

(a) [25%] Solve $y' = \dfrac{1+x^2}{1+x}$.

(b) [25%] Solve $y' = (\sin x + \cos x)(\sin x - \cos x)$.

(c) [25%] Solve $y' = x\tan\left(x^2\right)$, $y(0) = 2$.

(d) [25%] Find the position $x(t)$ from the velocity model $\dfrac{d}{dt}(tv) = 0$, $v(1) = 10$ and the position model $\dfrac{dx}{dt} = v(t)$, $x(1) = -1$.

(a) $$\begin{array}{r|l} & x-1 \\ \hline x+1 & x^2+1 \\ & x^2+x \\ \hline & -x+1 \\ & -x-1 \\ \hline & 2 \end{array}$$

$$y = c + \int \left(x - 1 + \frac{2}{x+1}\right)dx = c + \frac{x^2}{2} - x + 2\ln|1+x|$$

(b) $$y = c + \int u(-du) \qquad \left[\begin{array}{l} u = \sin x + \cos x \\ du = (\cos x - \sin x)\,dx \end{array}\right.$$

$$= c - \frac{u^2}{2}$$

$$= c - \frac{1}{2}(\sin x + \cos x)^2$$

(c) $$y = c + \frac{1}{2}\int \tan u\, du \qquad \left[\begin{array}{l} u = x^2 \\ du = 2x\,dx \end{array}\right.$$

$$= c + \frac{1}{2}\int -\frac{dw}{w} \qquad \left[\begin{array}{l} w = \cos u \\ dw = -\sin u\, du \end{array}\right.$$

$$= c - \frac{1}{2}\ln|\cos u| = c - \frac{1}{2}\ln|\cos(x^2)|$$

At $x=0$: $2 = c - \frac{1}{2}\ln|\cos 0| \Rightarrow c = 2$

$$y = 2 - \frac{1}{2}\ln|\cos x^2|$$

(d) $tv = c$

$1 \cdot 10 = c$ (Let $t=1$, $v=10$)

$$\boxed{v = \frac{10}{t}}$$

$$\left\{\begin{array}{l} x' = \dfrac{10}{t} \\ x(1) = -1 \end{array}\right.$$

$$\boxed{x = 10\ln|t| - 1}$$

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2. (Classification of Equations)
The differential equation $y' = f(x, y)$ is defined to be **separable** provided $f(x, y) = F(x)G(y)$ for some functions F and G.

(a) [40%] Check ($\boxed{X}$) the problems that can be put into separable form. No details expected.

$\boxtimes$	$y' + xy = y(2x + e^x) + x^2 y$	$\square$	$y' = (x-1)(y+1) - (x+1)y$
$\boxtimes$	$y' = 2e^{2x-y}e^{3y} + 3e^{3x+2y}$	$\square$	$y' + e^y = xy$

(b) [10%] Is $y' + xy = ye^x$ separable? No details expected. YES

(c) [10%] Give an example of $y' = f(x, y)$ which is separable and linear but not quadrature. No details expected.

(d) [40%] Apply tests to show that $y' = e^x + e^y$ is not separable and not linear. Supply all details.

(a) $y' + xy = 2xy + ye^x + x^2 y$ Linear
$y' = (x + e^x + x^2)y$ Separable

$y' = xy + x - y - 1 - xy - y$
$y' = x - 2y - 1$ Linear, Not Separable

$y' = 2e^{2x+2y} + 3e^{3x+2y}$
$y' = (2e^{2x} + 3e^{3x})e^{2y}$ Separable

$y' = -e^y + xy$ Not separable, Not Linear

(b) $y' = (-x + e^x)y$ Separable

(c) The first one in (a) works. Also $y' = xy$.

(d) Let $f(x,y) = e^x + e^y$. Then

$$\frac{f_x}{f} = \frac{e^x}{e^x + e^y}$$ depends on y $\Rightarrow$ $y' = e^x + e^y$ not separable.

$$\frac{\partial f}{\partial y} = e^y$$ not independent of y $\Rightarrow$ $y' = e^x + e^y$ not linear.

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3. (Solve a Separable Equation)
Given $(x+2)(y-1)y' = \left((x+2)e^{-x+2} + 2x^2 + 3\right)(y+1)(y-2)$.

Find a non-equilibrium solution in implicit form.
To save time, **do not solve** for y explicitly and **do not solve** for equilibrium solutions.

$$\frac{y-1}{(y+1)(y-2)}\, y' = e^{2-x} + \frac{2x^2+3}{x+2}$$

← separated form $\dfrac{y'}{G(y)} = F(x)$

$$\frac{y-1}{(y+1)(y-2)} = \frac{\left(\frac{-2}{-3}\right)}{y+1} + \frac{\left(\frac{1}{3}\right)}{y-2}$$

partial fractions using Heaviside coverup method

$$\int (\text{LHS})\,dx = \frac{2}{3}\ln|y+1| + \frac{1}{3}\ln|y-2|$$

Division Algorithm

$$\begin{array}{r} 2x-4 \\ x+2 \,\overline{)\, 2x^2+3} \\ \underline{2x^2+4x} \\ 3-4x \\ \underline{-8-4x} \\ 11 \end{array}$$

$$\frac{2x^2+3}{x+2} = 2x-4 + \frac{11}{x+2}$$

$$\int (\text{RHS})\,dx = x^2 - 4x + 11\ln|x+2| - e^{2-x}$$

answer in implicit form

$$\frac{2}{3}\ln|y+1| + \frac{1}{3}\ln|y-2| = -e^{2-x} + x^2 - 4x + 11\ln|x+2| + C$$

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4. (Linear Equations)

(a) [60%] Solve the linear model $5x'(t) = -160 + \frac{25}{2t+3}x(t)$, $x(0) = 32$. Show all integrating factor steps.

(b) [20%] Solve the homogeneous equation $\frac{dy}{dx} - (3x^2)y = 0$.

(c) [20%] Solve $2\frac{dy}{dx} + 5y = 1$ using the superposition principle $y = y_h + y_p$. Expected are answers for y_h and y_p.

(a) $x' - \frac{5}{2t+3}x = -\frac{160}{5}$

$\frac{(e^u x)'}{e^u} = -32$ || Replace LHS by the integrating factor Quotient

$(e^u x)' = -32e^u$

|| $u = \int \frac{-5}{2t+3}\,dt$

$e^u x = -32\int e^u dt$ || $u = -\frac{5}{2}\ln|2t+3|$

|| $e^u = (2t+3)^{-5/2}$

$e^u x = -32\int (2t+3)^{-5/2}dt$ use $x(0)=32$ to decide sign

$(2t+3)^{-5/2} x = -32\left(\frac{(2t+3)^{-3/2}}{(-3/2)2} + c\right)$ power rule used

$x(t) = \frac{32}{3}(2t+3) + \frac{32c}{3}(2t+3)^{5/2}$ ← Divide by e^u to isolate $x(t)$

$x(0)=32$ implies $c=0$

$$\boxed{x(t) = \frac{64}{3}t + 32}$$

(b) $y = \frac{c}{e^u}$ where $u = \int -3x^2\,dx = -x^3$

(c) $y_p = \frac{1}{5}$ $y_h = ce^{-5x/2}$

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5. (Stability)

(a) [50%] Draw a phase line diagram for the differential equation

$$\frac{dx}{dt} = \ln(1+4x^2)\,(3-|2x-1|)^3\,(2+x)(4-x^2)(1-x^2)^5.$$

Expected in the phase line diagram are equilibrium points and signs of dx/dt.

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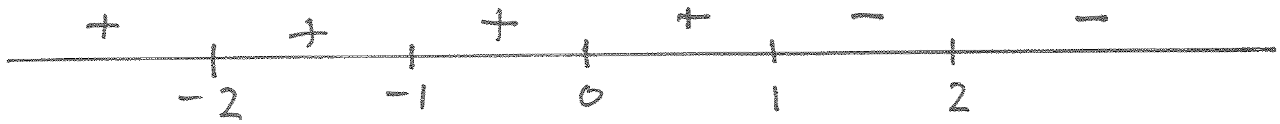
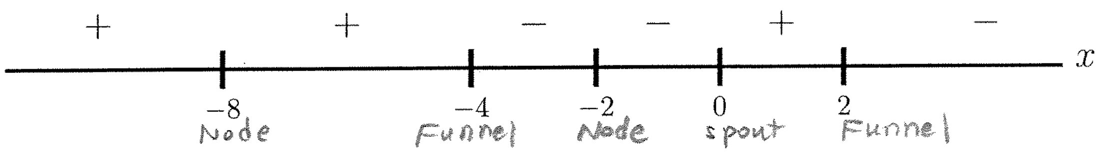
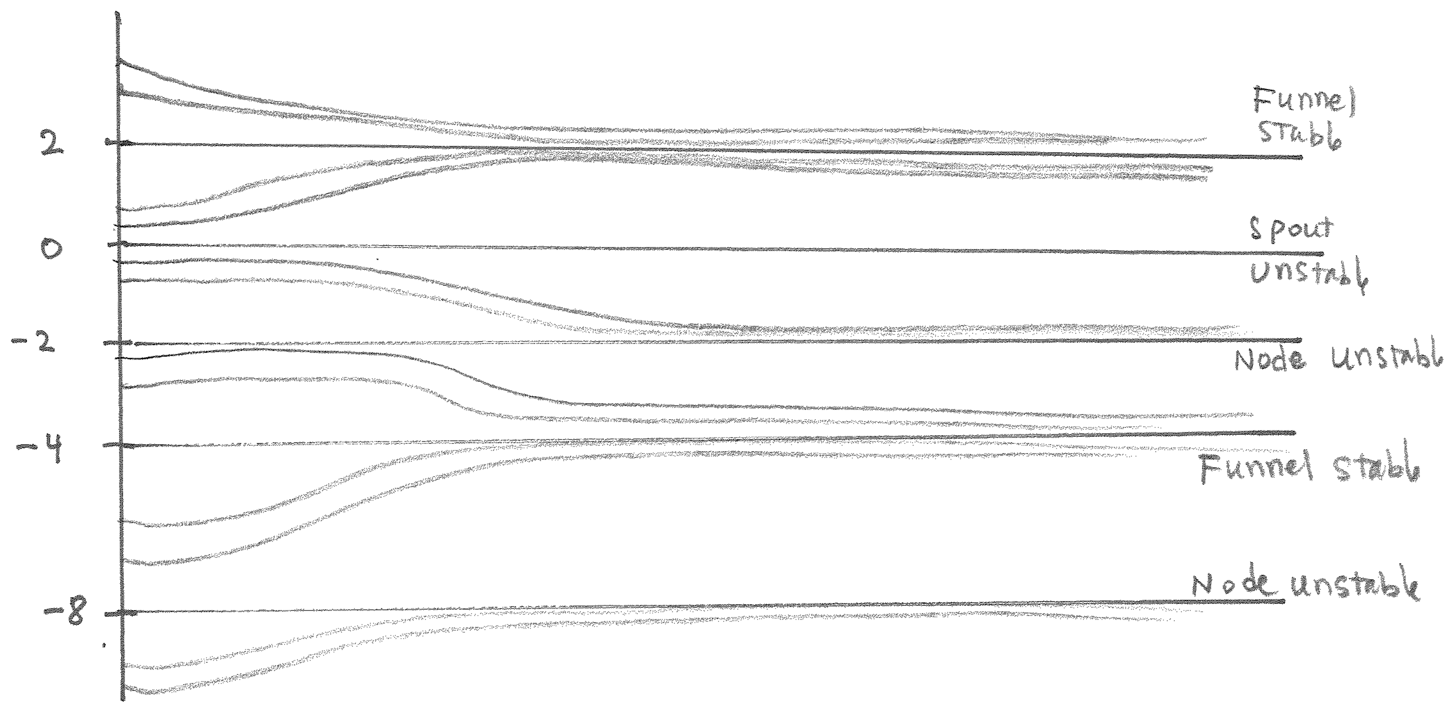
roots = 0, 2, −1, −2, 1

Handle $3-|2x-1|=0$ as 2 equations: $3-(2x-1)=0$ and $3+(2x-1)=0$



(b) [50%] Assume an autonomous equation $x'(t) = f(x(t))$. Draw a phase diagram with at least 12 threaded curves, using the phase line diagram given below. Add these labels as appropriate: funnel, spout, node [neither spout nor funnel], stable, unstable.





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