Vector Space V

It is a **data set** V plus a **toolkit** of eight (8) algebraic properties. The data set consists of packages of data items, called **vectors**, denoted $\vec{X}$.

Closure	The operations $\vec{X}+\vec{Y}$ and $k\vec{X}$ are defined and result in a new vector which is also in the set V.	
Addition	$\vec{X}+\vec{Y}=\vec{Y}+\vec{X}$	commutative
	$\vec{X}+(\vec{Y}+\vec{Z})=(\vec{Y}+\vec{X})+\vec{Z}$	associative
	Vector $\vec{0}$ is defined and $\vec{0}+\vec{X}=\vec{X}$	zero
	Vector $-\vec{X}$ is defined and $\vec{X}+(-\vec{X})=\vec{0}$	negative
Scalar multiply	$k(\vec{X}+\vec{Y})=k\vec{X}+k\vec{Y}$	distributive I
	$(k_1+k_2)\vec{X}=k_1\vec{X}+k_2\vec{X}$	distributive II
	$k_1(k_2\vec{X})=(k_1k_2)\vec{X}$	distributive III
	$1\vec{X}=\vec{X}$	identity

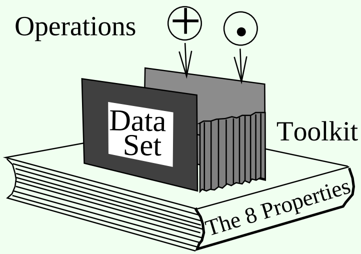

Figure 1. A *Vector Space* is a data storage system.

Definition. A **subspace** S of a vector space V is a nonvoid subset of V which under the operations $+$ and $\cdot$ of V forms a vector space in its own right. We call S a **working set**, because the purpose of identifying a subspace is to shrink the original data set V into a smaller data set S, customized for the application under study.

Subspaces, or working sets, are recognized as follows.

Subspace Criterion. Let S be a subset of V such that

1. Vector $\mathbf{0}$ is in S.

2. If $\vec{X}$ and $\vec{Y}$ are in S, then $\vec{X} + \vec{Y}$ is in S.

3. If $\vec{X}$ is in S, then $c\vec{X}$ is in S.

Then S is a subspace of V.

Items **2**, **3** can be summarized as *all linear combinations of vectors in* S *are again in* S.

Theorem 1 (Kernel Theorem)
Let V be one of the vector spaces R^n and let A be an $m \times n$ matrix. Define a smaller set S of data items in V by the kernel equation

$$S = \{\mathrm{x} \ : \ \mathrm{x} \text{ in } V, \quad A\mathrm{x} = 0\}.$$

Then S is a subspace of V.
In particular, operations of addition and scalar multiplication applied to data items in S give answers back in S, and the 8-property toolkit applies to data items in S.

Proof: Zero is in V because $A\mathrm{0} = \mathrm{0}$ for any matrix A. To verify the subspace criterion, we verify that $\mathrm{z} = c_1\mathrm{x} + c_2\mathrm{y}$ for x and y in V also belongs to V. The details:

$$\begin{aligned} A\mathrm{z} &= A(c_1\mathrm{x} + c_2\mathrm{y}) \\ &= A(c_1\mathrm{x}) + A(c_2\mathrm{y}) \\ &= c_1 A\mathrm{x} + c_2 A\mathrm{y} \\ &= c_1\mathrm{0} + c_2\mathrm{0} && \text{Because } A\mathrm{x} = A\mathrm{y} = \mathrm{0}, \text{ due to } \mathrm{x}, \mathrm{y} \text{ in } V. \\ &= \mathrm{0} && \text{Therefore, } A\mathrm{z} = \mathrm{0}, \text{ and } \mathrm{z} \text{ is in } V. \end{aligned}$$

The proof is complete.

Independence test for two vectors $\mathbf{v}_1, \mathbf{v}_2$

In an abstract vector space V, form the equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}.$$

Solve this equation for c_1, c_2. Then $\mathbf{v}_1, \mathbf{v}_2$ are independent in V if and only if the system has unique solution $c_1 = c_2 = 0$.

Illustration

Two column vectors are tested for independence by forming the system of equations $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}$, e.g,

$$c_1\begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This is a homogeneous system $A\mathbf{c} = \mathbf{0}$ with

$$A = \begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

The system $A\mathbf{c} = \mathbf{0}$ can be solved for $\mathbf{c}$ by $\mathbf{rref}$ methods. Because $\mathbf{rref}(A) = I$, then $c_1 = c_2 = 0$, which verifies independence.
If the system $A\mathbf{c} = \mathbf{0}$ is square, then $\det(A) \neq 0$ applies to test independence.
There is **no chance to use determinants** when the system is not square, e.g., consider the homogeneous system

$$c_1\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

It has vector-matrix form $A\mathbf{c} = \mathbf{0}$ with 3×2 matrix A, for which $\det(A)$ is undefined.

Rank Test

In the vector space $\boldsymbol{R}^n$, the key to detection of independence is **zero free variables**, or nullity zero, or equivalently, maximal rank. The test is justified from the formula $\mathbf{nullity}(\boldsymbol{A}) + \mathbf{rank}(\boldsymbol{A}) = \boldsymbol{k}$, where $\boldsymbol{k}$ is the column dimension of $\boldsymbol{A}$.

Theorem 2 (Rank-Nullity Test)

Let $\mathbf{v}_1, \ldots, \mathbf{v}_k$ be $\boldsymbol{k}$ column vectors in $\boldsymbol{R}^n$ and let $\boldsymbol{A}$ be the augmented matrix of these vectors. The vectors are independent if $\mathbf{rank}(\boldsymbol{A}) = \boldsymbol{k}$ and dependent if $\mathbf{rank}(\boldsymbol{A}) < \boldsymbol{k}$. The conditions are equivalent to $\mathbf{nullity}(\boldsymbol{A}) = 0$ and $\mathbf{nullity}(\boldsymbol{A}) > 0$, respectively.

Determinant Test

In the unusual case when the system arising in the independence test can be expressed as $A\mathbf{c} = \mathbf{0}$ and A is square, then $\det(A) = 0$ detects dependence, and $\det(A) \neq 0$ detects independence. The reasoning is based upon the adjugate formula $A^{-1} = \mathbf{adj}(A)/\det(A)$, valid exactly when $\det(A) \neq 0$.

Theorem 3 (Determinant Test)

Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be n column vectors in R^n and let A be the augmented matrix of these vectors. The vectors are independent if $\det(A) \neq 0$ and dependent if $\det(A) = 0$.

Not a Subspace Theorem

Theorem 4 (Testing S not a Subspace)
Let V be an abstract vector space and assume S is a subset of V. Then S is not a subspace of V provided one of the following holds.

(1) The vector $\mathbf{0}$ is not in S.

(2) Some $\mathbf{x}$ and $-\mathbf{x}$ are not both in S.

(3) Vector $\mathbf{x} + \mathbf{y}$ is not in S for some $\mathbf{x}$ and $\mathbf{y}$ in S.

Proof: The theorem is justified from the *Subspace Criterion*.
1. The criterion requires $\mathbf{0}$ is in S.
2. The criterion demands $c\mathbf{x}$ is in S for all scalars c and all vectors $\mathbf{x}$ in S.
3. According to the subspace criterion, the sum of two vectors in S must be in S.