Definitions

Pivot of A	A column in $\mathrm{rref}(A)$ which contains a leading one has a corresponding column in A, called a pivot column of A.
Basis of V	It is an independent set $\mathbf{v}_1, \ldots, \mathbf{v}_k$ from data set V whose linear combinations generate all data items in V. Generally, a basis is discovered by taking partial derivatives on symbols representing arbitrary constants.

Main Results

Theorem 1 (Dimension)

If a vector space V has a basis $\mathbf{v}_1, \ldots, \mathbf{v}_p$ and also a basis $\mathbf{u}_1, \ldots, \mathbf{u}_q$, then $p = q$. The **dimension** of V is this unique number p.

Theorem 2 (The Pivot Theorem)

- The pivot columns of a matrix A are linearly independent.
- A non-pivot column of A is a linear combination of the pivot columns of A.

Definitions

$\mathrm{rank}(A)$	The number of leading ones in $\mathrm{rref}(A)$
$\mathrm{nullity}(A)$	The number of columns of A minus $\mathrm{rank}(A)$

Main Results

Theorem 3 (Rank-Nullity Equation)
$\mathrm{rank}(A) + \mathrm{nullity}(A) =$ column dimension of A

Theorem 4 (Row Rank Equals Column Rank)
The number of independent rows of a matrix A equals the number of independent columns of A. Equivalently, $\mathrm{rank}(A) = \mathrm{rank}(A^T)$.

Theorem 5 (Pivot Method)
Let A be the augmented matrix of $\mathbf{v}_1, \ldots, \mathbf{v}_k$. Let the leading ones in $\mathrm{rref}(A)$ occur in columns $i_1, \ldots, i_p$. Then a largest independent subset of the k vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is the set

$$\mathbf{v}_{i_1}, \mathbf{v}_{i_2}, \ldots, \mathbf{v}_{i_p}.$$

Definitions

$\text{kernel}(A) = \text{nullspace}(A) = \{\mathbf{x} : A\mathbf{x} = \mathbf{0}\}$.
$\text{Image}(A) = \text{colspace}(A) = \{\mathbf{y} : \mathbf{y} = A\mathbf{x}$ for some $\mathbf{x}\}$.
$\text{rowspace}(A) = \text{colspace}(A^T) = \{\mathbf{w} : \mathbf{w} = A^T\mathbf{y}$ for some $\mathbf{y}\}$.
$\dim(V)$ is the number of elements in a basis for V.

How to Compute Null, Row, Column Space

Null Space. Compute $\mathbf{rref}(A)$. Write out the general solution $\mathbf{x}$ to $A\mathbf{x} = \mathbf{0}$, where the free variables are assigned parameter names $t_1, \ldots, t_k$. Report the basis for $\mathbf{nullspace}(A)$ as the list $\partial_{t_1}\mathbf{x}, \ldots, \partial_{t_k}\mathbf{x}$.

Column Space. Compute $\mathbf{rref}(A)$. Identify the pivot columns $i_1, \ldots, i_k$. Report the basis for $\mathbf{colspace}(A)$ as the list of columns $i_1, \ldots, i_k$ of A.

Row Space. Compute $\mathbf{rref}(A^T)$. Identify the lead variable columns $i_1, \ldots, i_k$. Report the basis for $\mathbf{rowspace}(A)$ as the list of rows $i_1, \ldots, i_k$ of A.

Alternatively, compute $\mathbf{rref}(A)$, then $\mathbf{rowspace}(A)$ has a *different* basis consisting of the list of nonzero rows of $\mathbf{rref}(A)$.

Theorem 6 (Dimension Identities)

(a) $\dim(\text{nullspace}(A)) = \dim(\text{kernel}(A)) = \text{nullity}(A)$

(b) $\dim(\text{colspace}(A)) = \dim(\text{Image}(A)) = \text{rank}(A)$

(c) $\dim(\text{rowspace}(A)) = \text{rank}(A)$

(d) $\dim(\text{kernel}(A)) + \dim(\text{Image}(A)) =$ column dimension of A

(e) $\dim(\text{kernel}(A)) + \dim(\text{kernel}(A^T)) =$ column dimension of A

Theorem 7 (Equivalence Test for Bases)
Define augmented matrices

$$B = \mathbf{aug}(\mathbf{v}_1, \ldots, \mathbf{v}_k), \quad C = \mathbf{aug}(\mathbf{u}_1, \ldots, \mathbf{u}_\ell), \quad W = \mathbf{aug}(B, C).$$

Then relation

$$k = \ell = \mathbf{rank}(B) = \mathbf{rank}(C) = \mathbf{rank}(W)$$

implies

1. $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is an independent set.
2. $\mathbf{u}_1, \ldots, \mathbf{u}_\ell$ is an independent set.
3. $\mathbf{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} = \mathbf{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_\ell\}$

In particular, $\mathbf{colspace}(B) = \mathbf{colspace}(C)$ and each set of vectors is an equivalent basis for this vector space.

Proof: Because $\mathbf{rank}(B) = k$, then the first k columns of W are independent. If some column of C is independent of the columns of B, then W would have $k + 1$ independent columns, which violates $k = \mathbf{rank}(W)$. Therefore, the columns of C are linear combinations of the columns of B. Then vector space $\mathbf{colspace}(C)$ is a subspace of vector space $\mathbf{colspace}(B)$. Because both vector spaces have dimension k, then $\mathbf{colspace}(B) = \mathbf{colspace}(C)$. The proof is complete.

Equivalent Bases: Computer Illustration

The following `maple` code applies the theorem to verify that two bases are equivalent:

1. The basis is determined from the `colspace` command in `maple`.
2. The basis is determined from the pivot columns of A.

In `maple`, the report of the column space basis is identical to the nonzero rows of $\mathbf{rref}(A^T)$.

```
with(linalg):
A:=matrix([[1,0,3],[3,0,1],[4,0,0]]);
colspace(A);            # Solve Ax=0, basis v1,v2 below
v1:=vector([2,0,-1]);v2:=vector([0,2,3]);
rref(A);                # Find the pivot cols=1,3
u1:=col(A,1); u2:=col(A,3); # pivot col basis
B:=augment(v1,v2); C:=augment(u1,u2);
W:=augment(B,C);
rank(B),rank(C),rank(W); # Test requires all equal 2
```

A False Test for Equivalent Bases

The relation

$$\mathbf{rref}(\boldsymbol{B}) = \mathbf{rref}(\boldsymbol{C})$$

holds for a substantial number of matrices $\boldsymbol{B}$ and $\boldsymbol{C}$. However, it does not imply that each column of $\boldsymbol{C}$ is a linear combination of the columns of $\boldsymbol{B}$. For example, define

$$\boldsymbol{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad \boldsymbol{C} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then

$$\mathbf{rref}(\boldsymbol{B}) = \mathbf{rref}(\boldsymbol{C}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix},$$

but $\mathbf{col}(\boldsymbol{C}, \mathbf{2})$ is not a linear combination of the columns of $\boldsymbol{B}$. This means $\mathbf{colspace}(\boldsymbol{B}) \neq \mathbf{colspace}(\boldsymbol{C})$.

Geometrically, the column spaces are planes in $\boldsymbol{R}^3$ which intersect only along the line $\boldsymbol{L}$ through the two points $(\mathbf{0}, \mathbf{0}, \mathbf{0})$ and $(\mathbf{1}, \mathbf{0}, \mathbf{1})$.