Linear Nonhomogeneous System ⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯

Given numbers a_{11}, ..., a_{mn}, b_1, ..., b_m, consider the **nonhomogeneous system** of m linear equations in n **unknowns** $x_1, x_2, \ldots, x_n$

(1)
$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \ &= \ b_1, \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \ &= \ b_2, \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \ &= \ b_m.
\end{aligned}
$$

Constants $a_{11}, \ldots, a_{mn}$ are called the **coefficients** of system (1). Constants b_1, ..., b_m are collectively referenced as the **right hand side**, **right side** or **RHS**.

Linear Homogeneous System ──────────────────────────────────

Given numbers $a_{11}, \ldots, a_{mn}$ consider the **homogeneous system** of m linear equations in n **unknowns** $x_1, x_2, \ldots, x_n$

(2)
$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0, \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0, \\
&\ \ \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0.
\end{aligned}
$$

Constants $a_{11}, \ldots, a_{mn}$ are called the **coefficients** of system (2).

The Three Possibilities

Solutions of general linear system (1) may be classified into exactly **three possibilities**:

> **1**. No solution.
> **2**. Infinitely many solutions.
> **3**. A unique solution.

Examples ───

A **solution** (x, y) of a 2×2 system is a pair of values that simultaneously satisfy both equations. Consider the following three systems:

1 No Solution

2 Infinitely Many Solutions

3 Unique Solution

$$\begin{cases} x - y = 0, \\ 0 = 1. \end{cases}$$

$$\begin{cases} x - 2y = 0, \\ 0 = 0. \end{cases}$$

$$\begin{cases} 3x + 2y = 1, \\ x - y = 2. \end{cases}$$

Explanations ───

- System **1** cannot have a solution because of the **signal equation $0 = 1$**, a false equation.

- System **2** has infinitely many solutions, one solution (x, y) for each point on the line $x - 2y = 0$. Analytic geometry writes the solutions for $-\infty < t_1 < \infty$ as the *parametric equations*

$$\begin{cases} x = 2t_1, \\ y = t_1. \end{cases}$$

- System **3** has a unique solution $x = 1, y = -1$.

Definition 1 (Parametric Equations)
The terminology **parametric equations** refers to a set of equations of the form

(3)
$$\begin{aligned}
x_1 &= d_1 + c_{11}t_1 + \cdots + c_{1k}t_k, \\
x_2 &= d_2 + c_{21}t_1 + \cdots + c_{2k}t_k, \\
&\vdots \\
x_n &= d_n + c_{n1}t_1 + \cdots + c_{nk}t_k.
\end{aligned}$$

The numbers $d_1, \ldots, d_n, c_{11}, \ldots, c_{nk}$ are *known constants* and the variable names $t_1, \ldots, t_k$ are **parameters**. The symbols $t_1, \ldots, t_k$ are therefore allowed to take on any value from $-\infty$ to ∞.

Analytic Geometry ───────────────────────────────

- In calculus courses, *parametric equations* are encountered as scalar equations of lines ($k = 1$) and planes ($k = 2$).

- If no symbols $t_1, t_2, \ldots$ appear, then the equations describe a *point*.

Definition 2 (General Solution)

A **general solution** of (1) is a set of parametric equations (3) plus two additional requirements:

(4)   Equations (3) satisfy (1) for all real values of $t_1, \ldots, t_k$.

(5)   Any solution of (1) can be obtained from (3) by specializing values of the parameter symbols $t_1, t_2, \ldots t_k$.

The Three Rules

The following rules *neither create nor destroy solutions* of the original system.

Swap Two equations can be interchanged without changing the solution set.

Mult An equation can be multiplied by $m \neq 0$ without changing the solution set.

Combo A multiple of one equation can be added to a different equation without changing the solution set.

Reversible Rules

The `mult` and `combo` rules replace an existing equation by a new one, whereas `swap` replaces two equations. The three operations are **reversible**:

- The `swap` rule is reversed by repeating it.

- The **mult** rule is reversed by repeating it with multiplier $1/m$.

- The **combo** rule is reversed by repeating it with c replaced by $-c$.

Reduced Echelon Systems ───────────────────────────────

- A **lead variable** is a variable that appears first (left-to-right) with coefficient one in exactly one equation.

- A system of linear algebraic equations in which each nonzero equation has a **lead variable** is called a **reduced echelon system**. The conventions:

  - Within an equation, variables must appear in variable list order.
  - Equations with lead variables are listed in variable list order.
  - Following them are any zero equations.

- A **free variable** in a reduced echelon system is any variable that is not a lead variable.

Recognition of Reduced Echelon Systems _____

A linear system (1) is recognized as a reduced echelon system exactly when the first variable listed in each equation has coefficient one and that variable name appears nowhere else in the system.

Form of a Reduced Echelon System _____
A reduced echelon system can be written in the special form

(6)
$$\begin{array}{rcl}
x_{i_1} + E_{11}x_{j_1} + \cdots + E_{1k}x_{j_k} & = & D_1, \\
x_{i_2} + E_{21}x_{j_1} + \cdots + E_{2k}x_{j_k} & = & D_2, \\
& \vdots & \\
x_{i_m} + E_{m1}x_{j_1} + \cdots + E_{mk}x_{j_k} & = & D_m.
\end{array}$$

Symbols Defined _____

- The numbers $E_{11}, \ldots, E_{mk}$ and $D_1, \ldots, D_m$ are *known constants*.

- Variables $x_{i_1}, \ldots, x_{i_m}$ are the lead variables.

- The remaining variables $x_{j_1}, \ldots, x_{i_k}$ are the free variables.

Writing a Standard Parametric Solution ⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯

Assume variable list order x, y, z, w, u, v for the **reduced echelon system** below. Boxed variables are lead variables and the remaining are free variables.

$$(7) \quad \begin{aligned} \boxed{x} + 4w + u + v &= 1, \\ \boxed{y} - u + v\phantom{{}+ 2u} &= 2, \\ \boxed{z} - w + 2u - v &= 0. \end{aligned}$$

General Solution Algorithm ⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯

If the reduced echelon system has zero free variables, then the unique solution is already displayed. Otherwise, there is at least one free variable, and then the 2–step algorithm below applies to write out the general solution.

1. Set the free variables equal to *invented symbols $t_1, \ldots, t_k$*. Each symbol can assume values from $-\infty$ to ∞.

2. Solve equations (7) for the leading variables and then back-substitute the free variables to obtain a **standard general solution**.

From Reduced Echelon System To General Solution _____

Assume variable list order x, y, z, w, u, v in the reduced echelon system

(8)
$$
\begin{aligned}
\boxed{x} + 4w + u + v &= 1, \\
\boxed{y} - u + v &= 2, \\
\boxed{z} - w + 2u - v &= 0.
\end{aligned}
$$

The boxed **lead variables** in (8) are x, y, z and the **free variables** are w, u, v. Assign invented symbols t_1, t_2, t_3 to the free variables and back-substitute in (8) to obtain a **standard general solution**

$$
\left\{
\begin{aligned}
x &= 1 - 4t_1 - t_2 - t_3, \\
y &= 2 + t_2 - t_3 \\
z &= t_1 - 2t_2 + t_3, \\
w &= t_1, \\
u &= t_2, \\
v &= t_3.
\end{aligned}
\right.
$$

Elimination Algorithm _____

The algorithm employs at each algebraic step one of the **three rules** defined previously as **mult**, **swap** and **combo**.

- The objective of each algebraic step is to **increase the number of lead variables**. The process stops when no more lead variables can be found, in which case the last system of equations is a **reduced echelon system**. It may also stop when a signal equation is found. Otherwise, equations with lead variables migrate to the top, in variable list order dictated by the lead variable, and equations with no variables are swapped to the end. Within each equation, variables appear in variable list order, left-to-right.

- Reversibility of the algebraic steps means that no solutions are created or destroyed: the original system and all systems in the intermediate steps have *exactly the same solutions*.

- The final reduced echelon system has a standard **general solution**. This expression is either the unique solution of the system, or else in the case of infinitely many solutions, it is a parametric solution using invented symbols $t_1, \ldots, t_k$.

Theorem 1 (Elimination)

Every linear system has either no solution or else it has exactly the same solutions as an equivalent reduced echelon system, obtained by repeated application of the three rules of **swap**, **mult** and **combo**.

Documenting the 3 Rules

Throughout, symbols `s` and `t` stand for `source` and `target` equations. Symbols `c` and `m` stand for `constant` and `multiplier`. The multiplier must be nonzero. The symbol R used in textbooks abbreviates **Row**, which corresponds exactly to **equation**.

Swap(s,t)	Interchange row s and row t.
	Textbooks: SWAP R_s, R_t
	Blackboard: `swap(s,t)`
Mult(t,m)	Multiply row t by $m \neq 0$.
	Textbooks: $(m)R_t$
	Blackboard: `mult(t,m)`
Combo(s,t,c)	Multiply row s by c and add to a different row t.
	Textbooks: $R_t = (c)R_s + R_t$
	Blackboard: `combo(s,t,c)`

From Equations to Matrices

The system $\begin{cases} 3x + 4y = 1 \\ 5x - 6y = 2 \end{cases}$ can be represented as the augmented matrix

$C = \begin{pmatrix} 3 & 4 & 1 \\ 5 & -6 & 2 \end{pmatrix}$. The process is automated for any number of variables, by observing that each column of C is a partial derivative, e.g., column 1 of C is the partial on x, column 2 of C the partial on y. The right column of C contains the RHS of each equation.

From Matrices to Equations

The variable names x, y are written above columns 1, 2 of the augmented matrix, as below.

$$\begin{array}{cc} x & y = \end{array}$$
$$\begin{pmatrix} 3 & 4 & \Big| & 1 \\ 5 & -6 & \Big| & 2 \end{pmatrix}$$

The scalar system is reconstructed from the augmented matrix by taking dot products, of the symbols on the top row against the rows of the matrix. Each dot product answer is followed by an equal sign and then the number in the third column.

The Three Rules and `Maple`

Newer versions of `maple` use the `LinearAlgebra` package, and a single function `RowOperation()` to accomplish the same thing done by three functions `addrow()`, `mulrow()`, `swaprow()` in the older `linalg` package. A conversion table appears below. This table can be used to create `maple` macros and functions for an easy conversion from hand-written text to `maple` code.

Hand-written	maple linalg	maple LinearAlgebra
`swap(s,t)`	`swaprow(A,s,t)`	`RowOperation(A,[t,s])`
`mult(t,c)`	`mulrow(A,t,c)`	`RowOperation(A,t,c)`
`combo(s,t,c)`	`addrow(A,s,t,c)`	`RowOperation(A,[t,s],c)`

RREF ———————————————————————————

The reduced row-echelon form of a matrix, or **rref**, is specified by the following requirements.

- Zero rows appear last.

- Each nonzero row has first element 1, called a **leading one**. The column in which the leading one appears, called a **pivot column**, has all other entries zero.

- The pivot columns appear as *consecutive* initial columns of the identity matrix I. Trailing columns of I might be absent.

RREF Illustration

The matrix (9) below is a typical **rref** which satisfies the preceding properties. Displayed secondly is the reduced echelon system (10) in the variables $x_1, \ldots, x_8$ represented by the augmented matrix (9).

(9)
$$
\left(
\begin{array}{cccccccc|c}
1 & 2 & 0 & 3 & 4 & 0 & 5 & 0 & 6 \\
0 & 0 & 1 & 7 & 8 & 0 & 9 & 0 & 10 \\
0 & 0 & 0 & 0 & 0 & 1 & 11 & 0 & 12 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 13 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
\right)
$$

(10)
$$
\begin{cases}
x_1 \;+2x_2 \qquad\; +3x_4 \;+4x_5 \qquad\; +\;5x_7 \qquad\quad = \;\; 6 \\
\qquad\qquad\quad x_3 \;+7x_4 \;+8x_5 \qquad\; +\;9x_7 \qquad\quad = 10 \\
\qquad\qquad\qquad\qquad\qquad\qquad x_6 \;+11x_7 \qquad\quad = 12 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\; x_8 \;=13 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\;\; 0 \;= \;\; 0 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\;\; 0 \;= \;\; 0 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\;\; 0 \;= \;\; 0
\end{cases}
$$

Matrix (9) is an **rref**, because it passes the two checkpoints. The pivot columns 1, 3, 6, 8 appear as the initial 4 columns of the 7×7 identity matrix I, in natural order; the trailing 3 columns of I are absent.

Frame Sequence Defined

- A sequence of swap, multiply and combination steps applied to a system of equations is called a **frame sequence**. Each step consists of exactly one operation.

- The **First Frame** is the original system and the **Last Frame** is the reduced echelon system.

- Frames are documented by the acronyms `swap, combo, mult`. Arithmetic detail is suppressed: only results appear.

- The *viewpoint* is that a camera is pointed over the shoulder of an assistant who writes the mathematics, and after the completion of each step, a photo is taken. The sequence of photo frames is the frame sequence. The photo sequence is related to a video of the events, because it is slide show of certain video frames.

Matrix Frame Sequences

The same terminology applies for systems $A\vec{x} = \vec{b}$ represented by an augmented matrix $C = \mathbf{aug}(A, \vec{b})$. The First Frame is C and the Last Frame is $\mathbf{rref}(C)$.

A Matrix Frame Sequence Illustration ————————————————

Steps in a frame sequence can be documented by the notation

```
swap(s,t), mult(t,m), combo(s,t,c),
```

each written next to the target row.

During the sequence, initial columns of the identity, called **pivot columns**, are created as steps toward the **rref**, using the three operations `mult, swap, combo`.

Trailing columns of the identity might not appear. Required is that pivot columns occur as consecutive initial columns of the identity matrix.

Frame 1: $\begin{pmatrix} 1 & 2 & -1 & 0 & 1 \\ 1 & 4 & -1 & 0 & 2 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ Original augmented matrix.

Frame 2: $\begin{pmatrix} 1 & 2 & -1 & 0 & 1 \\ 0 & 2 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ `combo(1,2,-1)`
Pivot column 1 completed.

Frame 3: $\begin{pmatrix} 1 & 2 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ `swap(2,3)`

Frame 4: $\begin{pmatrix} 1 & 2 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & -2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ `combo(2,3,-2)`

Frame 5: $\begin{pmatrix} 1 & 0 & -3 & 0 & -1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & -2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ `combo(2,1,-2)`
Pivot column 2 completed.

Frame 6: $\begin{pmatrix} 1 & 0 & -3 & 0 & -1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ `mult(3,-1/2)`

Frame 7: $\begin{pmatrix} 1 & 0 & -3 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1/2 \\ 0 & 0 & 1 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ `combo(3,2,-1)`

Last Frame: $\begin{pmatrix} 1 & 0 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 0 & 1/2 \\ 0 & 0 & 1 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ `combo(3,1,3)`
Pivot column 3 completed.
rref found.