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Definition 1 (Reduced Echelon System)
A linear system in which each nonzero equation has a **lead variable** is called a **reduced echelon system**.

Definition 2 (Rank and Nullity)
The number of lead variables in a reduced echelon system is called the **rank** of the system. The number of free variables in a reduced echelon system is called the **nullity** of the system.

We determine the **rank** and **nullity** of a system as follows. First, display a frame sequence which starts with that system and ends in a reduced echelon system. Then the rank and nullity of the system are those determined by the final frame.

Theorem 1 (Rank and Nullity)
The following equation holds:

$$\mathbf{rank} + \mathbf{nullity} = \text{number of variables}.$$

Elimination

The elimination algorithm applies at each algebraic step one of the three toolkit rules **swap**, **multiply** and **combination**.

- The objective of each algebraic step is to **increase the number of lead variables**. The process stops when a signal equation (typically $0 = 1$) is found. Otherwise, it stops when no more lead variables can be found, and then the last system of equations is a **reduced echelon system**. A detailed explanation of the process has been given in the discussion of frame sequences.

- Reversibility of the algebraic steps means that no solutions are created nor destroyed throughout the algebraic steps: the original system and all systems in the intermediate steps have *exactly the same solutions*.

- The final reduced echelon system has either a unique solution or infinitely many solutions. In both cases we report the **general solution**. In the infinitely many solution case, the **last frame algorithm** is used to write out a general solution.

Theorem 2 (Elimination)
Every linear system has either no solution or else it has exactly the same solutions as an equivalent reduced echelon system, obtained by repeated application of the toolkit rules **swap**, **multiply** and **combination**.

An Elimination Algorithm

An equation is said to be **processed** if it has a lead variable. Otherwise, the equation is said to be **unprocessed**.

1. If an equation "$0 = 0$" appears, then move it to the end. If a signal equation "$0 = c$" appears ($c \neq 0$ required), then the system is inconsistent. In this case, the algorithm halts and we report **no solution**.

2. Identify the **first symbol** x_r, in variable list order $x_1, \ldots, x_n$, which appears in some unprocessed equation. Apply the **multiply** rule to insure x_r has leading coefficient one. Apply the **combination** rule to eliminate variable x_r from all other equations. Then x_r is a **lead variable**: the number of lead variables has been increased by one.

3. Apply the **swap** rule repeatedly to move this equation past all processed equations, but before the unprocessed equations. Mark the equation as **processed**, e.g., replace x_r by boxed symbol $\boxed{x_r}$.

4. Repeat steps 1–3, until all equations have been processed once. Then lead variables $x_{i_1}, \ldots, x_{i_m}$ have been defined and the last system is a reduced echelon system.

1 Example (Elimination) Solve the system.

$$\begin{array}{rcrcrcrcr} w & + & 2x & - & y & + & z & = & 1, \\ w & + & 3x & - & y & + & 2z & = & 0, \\ & & x & & & + & z & = & -1. \end{array}$$

Solution

The answer using the natural variable list order w, x, y, z is the standard general solution

$$\begin{aligned} w &= 3 + t_1 + t_2, \\ x &= -1 - t_2, \\ y &= t_1, \\ z &= t_2, \qquad\qquad -\infty < t_1, t_2 < \infty. \end{aligned}$$

Details. Elimination will be applied to obtain a frame sequence whose last frame justifies the reported solution. The details amount to applying the three rules **swap**, **multiply** and **combination** for equivalent equations to obtain a last frame which is a reduced echelon system. The standard general solution for the last frame matches the one reported above. Let's mark processed equations with a box enclosing the lead variable ($\boldsymbol{w}$ is marked $\boxed{\boldsymbol{w}}$).

$$\begin{array}{|rcrcrcrcr|}\hline w &+& 2x &-& y &+& z &=& 1 \\ w &+& 3x &-& y &+& 2z &=& 0 \\ && x && &+& z &=& -1 \\ \hline\end{array} \qquad \boxed{\mathbf{1}}$$

$$\begin{array}{|rcrcrcrcr|}\hline w &+& 2x &-& y &+& z &=& 1 \\ 0 &+& x &+& 0 &+& z &=& -1 \\ && x && &+& z &=& -1 \\ \hline\end{array} \qquad \boxed{\mathbf{2}}$$

$$\begin{array}{|rcrcrcrcr|}\hline \boxed{w} &+& 2x &-& y &+& z &=& 1 \\ && x && &+& z &=& -1 \\ && && && 0 &=& 0 \\ \hline\end{array} \qquad \boxed{\mathbf{3}}$$

$$\begin{array}{|rcrcrcrcr|}\hline \boxed{w} &+& 0 &-& y &-& z &=& 3 \\ && \boxed{x} && &+& z &=& -1 \\ && && && 0 &=& 0 \\ \hline\end{array} \qquad \boxed{\mathbf{4}}$$

1 Original system. Identify the variable order as $\boldsymbol{w}$, $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$.

2 Choose $\boldsymbol{w}$ as a lead variable. Eliminate $\boldsymbol{w}$ from equation 2 by using `combo(1,2,-1)`.

3 The $\boldsymbol{w}$-equation is processed. Let $\boldsymbol{x}$ be the next lead variable. Eliminate $\boldsymbol{x}$ from equation 3 using `combo(2,3,-1)`.

4 Eliminate $\boldsymbol{x}$ from equation 1 using `combo(2,1,-2)`. Mark the $\boldsymbol{x}$-equation as processed. **Reduced echelon system** found.

The four frames make the **frame sequence** which takes the original system into a reduced echelon system. Basic exposition rules apply:

1. Variables in an equation appear in variable list order.
2. Equations inherit variable list order from the lead variables.

The last frame of the sequence, which must be a reduced echelon system, is used to write out the general solution, as follows.

$$\begin{array}{rcrcrcr} \boldsymbol{w} & = & \mathbf{3} & + & \boldsymbol{y} & + & \boldsymbol{z} \\ \boldsymbol{x} & = & \mathbf{-1} & - & \boldsymbol{z} & & \\ \boldsymbol{y} & = & \boldsymbol{t_1} & & & & \\ \boldsymbol{z} & = & \boldsymbol{t_2} & & & & \end{array}$$

Solve for the lead variables $\boldsymbol{w}$, $\boldsymbol{x}$. Assign invented symbols $\boldsymbol{t_1}$, $\boldsymbol{t_2}$ to the free variables $\boldsymbol{y}$, $\boldsymbol{z}$.

$$\begin{array}{rcrcrcr} \boldsymbol{w} & = & \mathbf{3} & + & \boldsymbol{t_1} & + & \boldsymbol{t_2} \\ \boldsymbol{x} & = & \mathbf{-1} & - & \boldsymbol{t_2} & & \\ \boldsymbol{y} & = & \boldsymbol{t_1} & & & & \\ \boldsymbol{z} & = & \boldsymbol{t_2} & & & & \end{array}$$

Back-substitute free variables into the lead variable equations to get a standard general solution.