

Determinant Theory

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Unique Solution of a 2×2 System

The 2×2 system

$$(1) \quad \begin{aligned} ax + by &= e, \\ cx + dy &= f, \end{aligned}$$

has a unique solution provided $\Delta = ad - bc$ is nonzero, in which case the solution is given by

$$(2) \quad x = \frac{de - bf}{ad - bc}, \quad y = \frac{af - ce}{ad - bc}.$$

This result is called **Cramer's Rule** for 2×2 systems, learned in college algebra.

Determinants of Order 2

College algebra introduces matrix notation and determinant notation:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

Evaluation of $\det(A)$ is by **Sarrus' 2×2 Rule**:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

The first product ad is the product of the main diagonal entries and the other product bc is from the anti-diagonal.

Cramer's 2×2 rule in determinant notation is

$$(3) \quad x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}.$$

Relation to Inverse Matrices

System

$$(4) \quad \begin{aligned} ax + by &= e, \\ cx + dy &= f, \end{aligned}$$

can be expressed as the vector-matrix system $\mathbf{A}\mathbf{u} = \mathbf{b}$ where

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} e \\ f \end{pmatrix}.$$

Inverse matrix theory implies

$$\mathbf{A}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \quad \mathbf{u} = \mathbf{A}^{-1}\mathbf{b} = \frac{1}{ad - bc} \begin{pmatrix} de - bf \\ af - ce \end{pmatrix}.$$

Cramer's Rule is a compact summary of the unique solution of system (4).

Unique Solution of an $n \times n$ System

System

$$(5) \quad \begin{array}{cccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1, \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2, \\ \vdots & & \vdots & & \cdots & & \vdots & & \vdots \\ a_{n1}x_1 & + & a_{n2}x_2 & + & \cdots & + & a_{nn}x_n & = & b_n \end{array}$$

can be written as an $n \times n$ vector-matrix equation $A\vec{x} = \vec{b}$, where $\vec{x} = (x_1, \dots, x_n)$ and $\vec{b} = (b_1, \dots, b_n)$. The system has a unique solution provided the **determinant of coefficients** $\Delta = \det(A)$ is nonzero, and then **Cramer's Rule** for $n \times n$ systems gives

$$(6) \quad x_1 = \frac{\Delta_1}{\Delta}, \quad x_2 = \frac{\Delta_2}{\Delta}, \quad \dots, \quad x_n = \frac{\Delta_n}{\Delta}.$$

Symbol $\Delta_j = \det(B)$, where matrix B has the same columns as matrix A , except $\text{col}(B, j) = \vec{b}$.

Determinants of Order n _____

Determinants will be defined shortly; intuition from the 2×2 case and Sarrus' rule should suffice for the moment.

Determinant Notation for Cramer's Rule

The **determinant of coefficients** for system $A\vec{x} = \vec{b}$ is denoted by

$$(7) \quad \Delta = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

The other n determinants in Cramer's rule (6) are given by

$$(8) \quad \Delta_1 = \begin{vmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ b_n & a_{n2} & \cdots & a_{nn} \end{vmatrix}, \dots, \Delta_n = \begin{vmatrix} a_{11} & a_{12} & \cdots & b_1 \\ a_{21} & a_{22} & \cdots & b_2 \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & b_n \end{vmatrix}.$$

College Algebra Definition of Determinant

Given an $n \times n$ matrix A , define

$$(9) \quad \det(A) = \sum_{\sigma \in S_n} (-1)^{\text{parity}(\sigma)} a_{1\sigma_1} \cdots a_{n\sigma_n}.$$

In the formula, a_{ij} denotes the element in row i and column j of the matrix A . The symbol $\sigma = (\sigma_1, \dots, \sigma_n)$ stands for a rearrangement of the subscripts $1, 2, \dots, n$ and S_n is the set of all possible rearrangements. The nonnegative integer $\text{parity}(\sigma)$ is determined by counting the minimum number of pairwise interchanges required to assemble the list of integers $\sigma_1, \dots, \sigma_n$ into natural order $1, \dots, n$.

College Algebra Definition and Sarrus' Rule

For a 3×3 matrix, the College Algebra formula reduces to **Sarrus' 3×3 Rule**

$$\begin{aligned} \det(A) &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ (10) \quad &= a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} \\ &\quad - a_{11}a_{32}a_{23} - a_{21}a_{12}a_{33} - a_{31}a_{22}a_{13}. \end{aligned}$$

Diagram for Sarrus' 3×3 Rule

The number $\det(\mathbf{A})$, in the 3×3 case, can be computed by the algorithm in Figure 1, which parallels the one for 2×2 matrices. The 5×3 array is made by copying the first two rows of \mathbf{A} into rows 4 and 5.

Warning: *there is no Sarrus' rule diagram for 4×4 or larger matrices!*

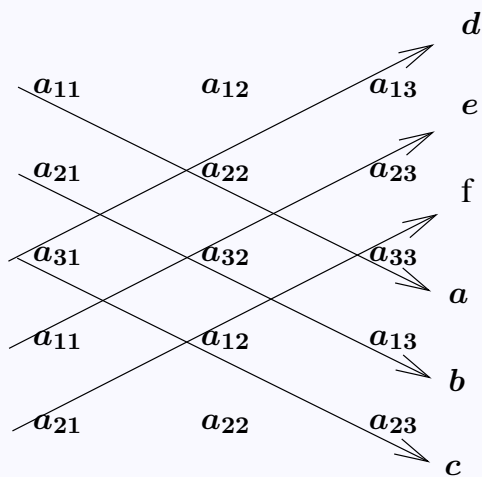


Figure 1. Sarrus' rule diagram for 3×3 matrices, which gives

$$\det(\mathbf{A}) = (a + b + c) - (d + e + f).$$

Transpose Rule

A consequence of the college algebra definition of determinant is the relation

$$\det(\mathbf{A}) = \det(\mathbf{A}^T)$$

where \mathbf{A}^T means the transpose of \mathbf{A} , obtained by swapping rows and columns. This relation implies the following.

All determinant theory results for rows also apply to columns.

How to Compute the Value of any Determinant _____

- **Four Rules.** These are the *Triangular Rule*, *Combination Rule*, *Multiply Rule* and the *Swap Rule*.
- **Special Rules.** These apply to evaluate a determinant as zero.
- **Cofactor Expansion.** This is an iterative scheme which reduces computation of a determinant to a number of smaller determinants.
- **Hybrid Method.** The four rules and the cofactor expansion are combined.

Four Rules

Triangular The value of $\det(\mathbf{A})$ for either an upper triangular or a lower triangular matrix \mathbf{A} is the product of the diagonal elements:

$$\det(\mathbf{A}) = a_{11}a_{22} \cdots a_{nn}.$$

This is a one-arrow Sarrus' rule.

Swap If \mathbf{B} results from \mathbf{A} by swapping two rows, then

$$\det(\mathbf{A}) = (-1) \det(\mathbf{B}).$$

Combination The value of $\det(\mathbf{A})$ is unchanged by adding a multiple of a row to a different row.

Multiply If one row of \mathbf{A} is multiplied by constant c to create matrix \mathbf{B} , then

$$\det(\mathbf{B}) = c \det(\mathbf{A}).$$

1 Example (Four Properties) Apply the four properties of a determinant to justify the formula

$$\det \begin{pmatrix} 12 & 6 & 0 \\ 11 & 5 & 1 \\ 10 & 2 & 2 \end{pmatrix} = 24.$$

Solution: Let D denote the value of the determinant. Then

$$D = \det \begin{pmatrix} 12 & 6 & 0 \\ 11 & 5 & 1 \\ 10 & 2 & 2 \end{pmatrix}$$

Given.

$$= \det \begin{pmatrix} 12 & 6 & 0 \\ -1 & -1 & 1 \\ -2 & -4 & 2 \end{pmatrix}$$

combo $(1, 2, -1)$, combo $(1, 3, -1)$. Combination leaves the determinant unchanged.

$$= 6 \det \begin{pmatrix} 2 & 1 & 0 \\ -1 & -1 & 1 \\ -2 & -4 & 2 \end{pmatrix}$$

Multiply rule $m = 1/6$ on row 1 factors out a 6.

$$= 6 \det \begin{pmatrix} 0 & -1 & 2 \\ -1 & -1 & 1 \\ 0 & -3 & 2 \end{pmatrix}$$

combo $(1, 3, 1)$, combo $(2, 1, 2)$.

$$= -6 \det \begin{pmatrix} -1 & -1 & 1 \\ 0 & -1 & 2 \\ 0 & -3 & 2 \end{pmatrix}$$

swap $(1, 2)$. Swap changes the sign of the determinant.

$$= 6 \det \begin{pmatrix} 1 & 1 & -1 \\ 0 & -1 & 2 \\ 0 & -3 & 2 \end{pmatrix}$$

Multiply rule $m = -1$ on row 1.

$$= 6 \det \begin{pmatrix} 1 & 1 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & -4 \end{pmatrix}$$

combo $(2, 3, -3)$.

$$= 6(1)(-1)(-4) = 24$$

Triangular rule. Formula verified.

Elementary Matrices and the Four Rules

The four rules can be stated in terms of elementary matrices as follows.

- Triangular** The value of $\det(\mathbf{A})$ for either an upper triangular or a lower triangular matrix \mathbf{A} is the product of the diagonal elements: $\det(\mathbf{A}) = a_{11}a_{22} \cdots a_{nn}$. This is a one-arrow Sarrus' rule valid for dimension n .
- Swap** If \mathbf{E} is an elementary matrix for a swap rule, then $\det(\mathbf{EA}) = (-1) \det(\mathbf{A})$.
- Combination** If \mathbf{E} is an elementary matrix for a combination rule, then $\det(\mathbf{EA}) = \det(\mathbf{A})$.
- Multiply** If \mathbf{E} is an elementary matrix for a multiply rule with multiplier $m \neq 0$, then $\det(\mathbf{EA}) = m \det(\mathbf{A})$.

Because $\det(\mathbf{E}) = 1$ for a combination rule, $\det(\mathbf{E}) = -1$ for a swap rule and $\det(\mathbf{E}) = c$ for a multiply rule with multiplier $c \neq 0$, it follows that for any elementary matrix \mathbf{E} there is the **determinant multiplication rule**

$$\det(\mathbf{EA}) = \det(\mathbf{E}) \det(\mathbf{A}).$$

Special Determinant Rules

The results are stated for rows but also hold for columns, because $\det(A) = \det(A^T)$.

| | |
|----------------|---|
| Zero row | If one row of A is zero, then $\det(A) = 0$. |
| Duplicate rows | If two rows of A are identical, then $\det(A) = 0$. |
| RREF $\neq I$ | If $\text{rref}(A) \neq I$, then $\det(A) = 0$. |
| Common factor | The relation $\det(A) = c \det(B)$ holds, provided A and B differ only in one row, say row j , for which $\text{row}(A, j) = c \text{row}(B, j)$. |
| Row linearity | The relation $\det(A) = \det(B) + \det(C)$ holds, provided A , B and C differ only in one row, say row j , for which $\text{row}(A, j) = \text{row}(B, j) + \text{row}(C, j)$. |

Cofactor Expansion for 3×3 Matrices

This is a review the college algebra topic, where the dimension of A is 3 .

Cofactor row expansion means the following formulas are valid:

$$\begin{aligned} |A| &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ &= a_{11}(+1) \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}(-1) \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}(+1) \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ &= a_{21}(-1) \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{22}(+1) \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + a_{23}(-1) \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \\ &= a_{31}(+1) \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} + a_{32}(-1) \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} + a_{33}(+1) \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{aligned}$$

The formulas expand a 3×3 determinant in terms of 2×2 determinants, along a row of A . The attached signs ± 1 are called the **checkerboard signs**, to be defined shortly. The 2×2 determinants are called **minors** of the 3×3 determinant $|A|$. The checkerboard sign together with a minor is called a **cofactor**.

Cofactor Expansion Illustration

Cofactor expansion formulas are generally used when a row has one or two zeros, making it unnecessary to evaluate one or two of the 2×2 determinants in the expansion. To illustrate, row 1 cofactor expansion gives

$$\begin{aligned} \begin{vmatrix} 3 & 0 & 0 \\ 2 & 1 & 7 \\ 5 & 4 & 8 \end{vmatrix} &= 3(+1) \begin{vmatrix} 1 & 7 \\ 4 & 8 \end{vmatrix} + 0(-1) \begin{vmatrix} 2 & 7 \\ 5 & 8 \end{vmatrix} + 0(+1) \begin{vmatrix} 2 & 1 \\ 5 & 4 \end{vmatrix} \\ &= 3(+1)(8 - 28) + 0 + 0 \\ &= -60. \end{aligned}$$

What has been said for rows also applies to columns, due to the transpose formula

$$\det(\mathbf{A}) = \det(\mathbf{A}^T).$$

Minor

The $(n - 1) \times (n - 1)$ determinant obtained from $\det(\mathbf{A})$ by striking out row i and column j is called the (i, j) -minor of \mathbf{A} and denoted $\mathbf{minor}(\mathbf{A}, i, j)$. Literature might use M_{ij} for a minor.

Cofactor

The (i, j) -cofactor of \mathbf{A} is $\mathbf{cof}(\mathbf{A}, i, j) = (-1)^{i+j} \mathbf{minor}(\mathbf{A}, i, j)$.

Multiplicative factor $(-1)^{i+j}$ is called the **checkerboard sign**, because its value can be determined by counting *plus, minus, plus*, etc., from location $(1, 1)$ to location (i, j) in any checkerboard fashion.

Expansion of Determinants by Cofactors

$$(11) \quad \det(\mathbf{A}) = \sum_{j=1}^n a_{kj} \mathbf{cof}(\mathbf{A}, k, j), \quad \det(\mathbf{A}) = \sum_{i=1}^n a_{i\ell} \mathbf{cof}(\mathbf{A}, i, \ell),$$

In (11), $1 \leq k \leq n$, $1 \leq \ell \leq n$. The first expansion is called a **cofactor row expansion** and the second is called a **cofactor column expansion**. The value $\mathbf{cof}(\mathbf{A}, i, j)$ is the cofactor of element a_{ij} in $\det(\mathbf{A})$, that is, the checkerboard sign times the minor of a_{ij} .

2 Example (Hybrid Method) Justify by cofactor expansion and the four properties the identity

$$\det \begin{pmatrix} 10 & 5 & 0 \\ 11 & 5 & a \\ 10 & 2 & b \end{pmatrix} = 5(6a - b).$$

Solution: Let D denote the value of the determinant. Then

$$D = \det \begin{pmatrix} 10 & 5 & 0 \\ 11 & 5 & a \\ 10 & 2 & b \end{pmatrix}$$

Given.

$$= \det \begin{pmatrix} 10 & 5 & 0 \\ 1 & 0 & a \\ 0 & -3 & b \end{pmatrix}$$

Combination leaves the determinant unchanged:
combo $(1, 2, -1)$, combo $(1, 3, -1)$.

$$= \det \begin{pmatrix} 0 & 5 & -10a \\ 1 & 0 & a \\ 0 & -3 & b \end{pmatrix}$$

combo $(2, 1, -10)$.

$$= (1)(-1) \det \begin{pmatrix} 5 & -10a \\ -3 & b \end{pmatrix}$$

Cofactor expansion on column 1.

$$= (1)(-1)(5b - 30a)$$

Sarrus' rule for $n = 2$.

$$= 5(6a - b).$$

Formula verified.

3 Example (Cramer's Rule) Solve by Cramer's rule the system of equations

$$\begin{aligned}2x_1 + 3x_2 + x_3 - x_4 &= 1, \\x_1 + x_2 - x_4 &= -1, \\3x_2 + x_3 + x_4 &= 3, \\x_1 + x_3 - x_4 &= 0,\end{aligned}$$

verifying $x_1 = 1$, $x_2 = 0$, $x_3 = 1$, $x_4 = 2$.

Solution: Form the four determinants $\Delta_1, \dots, \Delta_4$ from the base determinant Δ as follows:

$$\Delta = \det \begin{pmatrix} 2 & 3 & 1 & -1 \\ 1 & 1 & 0 & -1 \\ 0 & 3 & 1 & 1 \\ 1 & 0 & 1 & -1 \end{pmatrix},$$

$$\Delta_1 = \det \begin{pmatrix} 1 & 3 & 1 & -1 \\ -1 & 1 & 0 & -1 \\ 3 & 3 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \quad \Delta_2 = \det \begin{pmatrix} 2 & 1 & 1 & -1 \\ 1 & -1 & 0 & -1 \\ 0 & 3 & 1 & 1 \\ 1 & 0 & 1 & -1 \end{pmatrix},$$

$$\Delta_3 = \det \begin{pmatrix} 2 & 3 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 0 & 3 & 3 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix}, \quad \Delta_4 = \det \begin{pmatrix} 2 & 3 & 1 & 1 \\ 1 & 1 & 0 & -1 \\ 0 & 3 & 1 & 3 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

Five repetitions of the methods used in the previous examples give the answers $\Delta = -2$, $\Delta_1 = -2$, $\Delta_2 = 0$, $\Delta_3 = -2$, $\Delta_4 = -4$, therefore Cramer's rule implies the solution

$$x_1 = \frac{\Delta_1}{\Delta}, \quad x_2 = \frac{\Delta_2}{\Delta}, \quad x_3 = \frac{\Delta_3}{\Delta}, \quad x_4 = \frac{\Delta_4}{\Delta}.$$

Then $x_1 = 1$, $x_2 = 0$, $x_3 = 1$, $x_4 = 2$.

Maple Code for Cramer's Rule

The details of the computation above can be checked in computer algebra system maple as follows.

```
with(linalg):  
A:=matrix([  
  [2, 3, 1, -1], [1, 1, 0, -1],  
  [0, 3, 1, 1], [1, 0, 1, -1]]);  
Delta:= det(A);  
b:=vector([1,-1,3,0]):  
B1:=A: col(B1,1):=b:  
Delta1:=det(B1);  
x[1]:=Delta1/Delta;
```


The Adjugate Matrix

The **adjugate** $\mathbf{adj}(A)$ of an $n \times n$ matrix A is the transpose of the matrix of cofactors,

$$\mathbf{adj}(A) = \begin{pmatrix} \mathbf{cof}(A, 1, 1) & \mathbf{cof}(A, 1, 2) & \cdots & \mathbf{cof}(A, 1, n) \\ \mathbf{cof}(A, 2, 1) & \mathbf{cof}(A, 2, 2) & \cdots & \mathbf{cof}(A, 2, n) \\ \vdots & \vdots & \cdots & \vdots \\ \mathbf{cof}(A, n, 1) & \mathbf{cof}(A, n, 2) & \cdots & \mathbf{cof}(A, n, n) \end{pmatrix}^T.$$

A cofactor $\mathbf{cof}(A, i, j)$ is the checkerboard sign $(-1)^{i+j}$ times the corresponding minor determinant $\mathbf{minor}(A, i, j)$.

Adjugate of a 2×2

$$\mathbf{adj} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

In words: *swap the diagonal elements and change the sign of the off-diagonal elements.*

Adjugate Formula for the Inverse

For any $n \times n$ matrix

$$\mathbf{A} \cdot \mathbf{adj}(\mathbf{A}) = \mathbf{adj}(\mathbf{A}) \cdot \mathbf{A} = \det(\mathbf{A}) \mathbf{I}.$$

The equation is valid even if \mathbf{A} is not invertible. The relation suggests several ways to find $\det(\mathbf{A})$ from \mathbf{A} and $\mathbf{adj}(\mathbf{A})$ with one dot product.

For an invertible matrix \mathbf{A} , the relation implies $\mathbf{A}^{-1} = \mathbf{adj}(\mathbf{A}) / \det(\mathbf{A})$:

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} \begin{pmatrix} \mathbf{cof}(\mathbf{A}, 1, 1) & \mathbf{cof}(\mathbf{A}, 1, 2) & \cdots & \mathbf{cof}(\mathbf{A}, 1, n) \\ \mathbf{cof}(\mathbf{A}, 2, 1) & \mathbf{cof}(\mathbf{A}, 2, 2) & \cdots & \mathbf{cof}(\mathbf{A}, 2, n) \\ \vdots & \vdots & \cdots & \vdots \\ \mathbf{cof}(\mathbf{A}, n, 1) & \mathbf{cof}(\mathbf{A}, n, 2) & \cdots & \mathbf{cof}(\mathbf{A}, n, n) \end{pmatrix}^T$$

Elementary Matrices

Theorem 1 (Determinants and Elementary Matrices)

Let E be an $n \times n$ elementary matrix. Then

Combination $\det(E) = 1$

Multiply $\det(E) = m$ for multiplier m .

Swap $\det(E) = -1$

Product $\det(EX) = \det(E) \det(X)$ for all $n \times n$ matrices X .

Theorem 2 (Determinants and Invertible Matrices)

Let A be a given invertible matrix. Then

$$\det(A) = \frac{(-1)^s}{m_1 m_2 \cdots m_r}$$

where s is the number of swap rules applied and m_1, m_2, \dots, m_r are the nonzero multipliers used in multiply rules when A is reduced to $\text{rref}(A)$.

Determinant Products

Theorem 3 (Determinant Product Rule)

Let A and B be given $n \times n$ matrices. Then

$$\det(AB) = \det(A) \det(B).$$

Proof

Assume one of A or B has zero determinant. Then $\det(A) \det(B) = 0$. If $\det(B) = 0$, then $B\mathbf{x} = \mathbf{0}$ has infinitely many solutions, in particular a nonzero solution \mathbf{x} . Multiply $B\mathbf{x} = \mathbf{0}$ by A , then $AB\mathbf{x} = \mathbf{0}$ which implies AB is not invertible. Then the identity $\det(AB) = \det(A) \det(B)$ holds, because both sides are zero. If $\det(B) \neq 0$ but $\det(A) = 0$, then there is a nonzero \mathbf{y} with $A\mathbf{y} = \mathbf{0}$. Define $\mathbf{x} = AB^{-1}\mathbf{y}$. Then $AB\mathbf{x} = A\mathbf{y} = \mathbf{0}$, with $\mathbf{x} \neq \mathbf{0}$, which implies the identity holds.. This completes the proof when one of A or B is not invertible.

Assume A , B are invertible and then $C = AB$ is invertible. In particular, $\text{rref}(A^{-1}) = \text{rref}(B^{-1}) = I$. Write $I = \text{rref}(A^{-1}) = E_1 E_2 \cdots E_k A^{-1}$ and $I = \text{rref}(B^{-1}) = F_1 F_2 \cdots F_m B^{-1}$ for elementary matrices E_i, F_j . Then

$$(12) \quad AB = E_1 E_2 \cdots E_k F_1 F_2 \cdots F_m.$$

The theorem follows from repeated application of the basic identity $\det(EX) = \det(E) \det(X)$ to relation (12), because

$$\det(A) = \det(E_1) \cdots \det(E_k), \quad \det(B) = \det(F_1) \cdots \det(F_m).$$

