

## Determinant Theory

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## Unique Solution of a $2 \times 2$ System

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The  $2 \times 2$  system

$$(1) \quad \begin{aligned} ax + by &= e, \\ cx + dy &= f, \end{aligned}$$

has a unique solution provided  $\Delta = ad - bc$  is nonzero, in which case the solution is given by

$$(2) \quad x = \frac{de - bf}{ad - bc}, \quad y = \frac{af - ce}{ad - bc}.$$

This result is called **Cramer's Rule** for  $2 \times 2$  systems, learned in college algebra.

## Determinants of Order 2

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College algebra introduces matrix notation and determinant notation:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

Evaluation of  $\det(A)$  is by **Sarrus'  $2 \times 2$  Rule**:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

The first product  $ad$  is the product of the main diagonal entries and the other product  $bc$  is from the anti-diagonal.

Cramer's  $2 \times 2$  rule in determinant notation is

$$(3) \quad x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}.$$

## Relation to Inverse Matrices

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System

$$(4) \quad \begin{aligned} ax + by &= e, \\ cx + dy &= f, \end{aligned}$$

can be expressed as the vector-matrix system  $\mathbf{A}\mathbf{u} = \mathbf{b}$  where

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} e \\ f \end{pmatrix}.$$

Inverse matrix theory implies

$$\mathbf{A}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \quad \mathbf{u} = \mathbf{A}^{-1}\mathbf{b} = \frac{1}{ad - bc} \begin{pmatrix} de - bf \\ af - ce \end{pmatrix}.$$

Cramer's Rule is a compact summary of the unique solution of system (4).

## Unique Solution of an $n \times n$ System

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System

$$(5) \quad \begin{array}{cccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1, \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2, \\ \vdots & & \vdots & & \cdots & & \vdots & & \vdots \\ a_{n1}x_1 & + & a_{n2}x_2 & + & \cdots & + & a_{nn}x_n & = & b_n \end{array}$$

can be written as an  $n \times n$  vector-matrix equation  $A\vec{x} = \vec{b}$ , where  $\vec{x} = (x_1, \dots, x_n)$  and  $\vec{b} = (b_1, \dots, b_n)$ . The system has a unique solution provided the **determinant of coefficients**  $\Delta = \det(A)$  is nonzero, and then **Cramer's Rule** for  $n \times n$  systems gives

$$(6) \quad x_1 = \frac{\Delta_1}{\Delta}, \quad x_2 = \frac{\Delta_2}{\Delta}, \quad \dots, \quad x_n = \frac{\Delta_n}{\Delta}.$$

Symbol  $\Delta_j = \det(B)$ , where matrix  $B$  has the same columns as matrix  $A$ , except  $\text{col}(B, j) = \vec{b}$ .

## **Determinants of Order $n$** \_\_\_\_\_

Determinants will be defined shortly; intuition from the  $2 \times 2$  case and Sarrus' rule should suffice for the moment.

## Determinant Notation for Cramer's Rule

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The **determinant of coefficients** for system  $A\vec{x} = \vec{b}$  is denoted by

$$(7) \quad \Delta = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

The other  $n$  determinants in Cramer's rule (6) are given by

$$(8) \quad \Delta_1 = \begin{vmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ b_n & a_{n2} & \cdots & a_{nn} \end{vmatrix}, \dots, \Delta_n = \begin{vmatrix} a_{11} & a_{12} & \cdots & b_1 \\ a_{21} & a_{22} & \cdots & b_2 \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & b_n \end{vmatrix}.$$

## College Algebra Definition of Determinant

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Given an  $n \times n$  matrix  $A$ , define

$$(9) \quad \det(A) = \sum_{\sigma \in S_n} (-1)^{\text{parity}(\sigma)} a_{1\sigma_1} \cdots a_{n\sigma_n}.$$

In the formula,  $a_{ij}$  denotes the element in row  $i$  and column  $j$  of the matrix  $A$ . The symbol  $\sigma = (\sigma_1, \dots, \sigma_n)$  stands for a rearrangement of the subscripts  $1, 2, \dots, n$  and  $S_n$  is the set of all possible rearrangements. The nonnegative integer  $\text{parity}(\sigma)$  is determined by counting the minimum number of pairwise interchanges required to assemble the list of integers  $\sigma_1, \dots, \sigma_n$  into natural order  $1, \dots, n$ .

## College Algebra Definition and Sarrus' Rule

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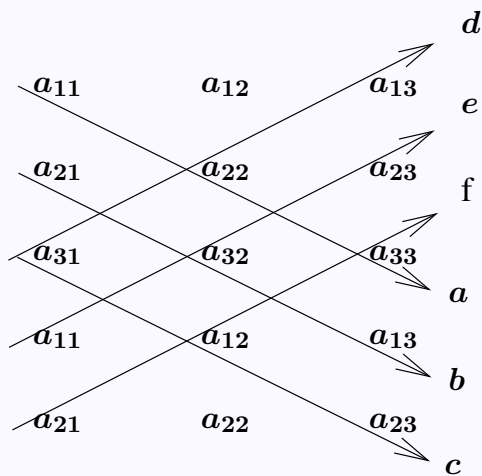
For a  $3 \times 3$  matrix, the College Algebra formula reduces to **Sarrus'  $3 \times 3$  Rule**

$$\begin{aligned} (10) \quad \det(A) &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ &= a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} \\ &\quad - a_{11}a_{32}a_{23} - a_{21}a_{12}a_{33} - a_{31}a_{22}a_{13}. \end{aligned}$$

## Diagram for Sarrus' $3 \times 3$ Rule

The number  $\det(\mathbf{A})$ , in the  $3 \times 3$  case, can be computed by the algorithm in Figure 1, which parallels the one for  $2 \times 2$  matrices. The  $5 \times 3$  array is made by copying the first two rows of  $\mathbf{A}$  into rows 4 and 5.

**Warning:** *there is no Sarrus' rule diagram for  $4 \times 4$  or larger matrices!*



**Figure 1.** Sarrus' rule diagram for  $3 \times 3$  matrices, which gives

$$\det(\mathbf{A}) = (a + b + c) - (d + e + f).$$

## Transpose Rule

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A consequence of the college algebra definition of determinant is the relation

$$\det(\mathbf{A}) = \det(\mathbf{A}^T)$$

where  $\mathbf{A}^T$  means the transpose of  $\mathbf{A}$ , obtained by swapping rows and columns. This relation implies the following.

All determinant theory results for rows also apply to columns.

## How to Compute the Value of any Determinant \_\_\_\_\_

- **Four Rules.** These are the *Triangular Rule*, *Combination Rule*, *Multiply Rule* and the *Swap Rule*.
- **Special Rules.** These apply to evaluate a determinant as zero.
- **Cofactor Expansion.** This is an iterative scheme which reduces computation of a determinant to a number of smaller determinants.
- **Hybrid Method.** The four rules and the cofactor expansion are combined.

## Four Rules

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**Triangular** The value of  $\det(\mathbf{A})$  for either an upper triangular or a lower triangular matrix  $\mathbf{A}$  is the product of the diagonal elements:

$$\det(\mathbf{A}) = a_{11}a_{22} \cdots a_{nn}.$$

This is a one-arrow Sarrus' rule.

**Swap** If  $\mathbf{B}$  results from  $\mathbf{A}$  by swapping two rows, then

$$\det(\mathbf{A}) = (-1) \det(\mathbf{B}).$$

**Combination** The value of  $\det(\mathbf{A})$  is unchanged by adding a multiple of a row to a different row.

**Multiply** If one row of  $\mathbf{A}$  is multiplied by constant  $c$  to create matrix  $\mathbf{B}$ , then

$$\det(\mathbf{B}) = c \det(\mathbf{A}).$$

**1 Example (Four Properties)** Apply the four properties of a determinant to justify the formula

$$\det \begin{pmatrix} 12 & 6 & 0 \\ 11 & 5 & 1 \\ 10 & 2 & 2 \end{pmatrix} = 24.$$

**Solution:** Let  $D$  denote the value of the determinant. Then

$$D = \det \begin{pmatrix} 12 & 6 & 0 \\ 11 & 5 & 1 \\ 10 & 2 & 2 \end{pmatrix}$$

Given.

$$= \det \begin{pmatrix} 12 & 6 & 0 \\ -1 & -1 & 1 \\ -2 & -4 & 2 \end{pmatrix}$$

combo  $(1, 2, -1)$ , combo  $(1, 3, -1)$ . Combination leaves the determinant unchanged.

$$= 6 \det \begin{pmatrix} 2 & 1 & 0 \\ -1 & -1 & 1 \\ -2 & -4 & 2 \end{pmatrix}$$

Multiply rule  $m = 1/6$  on row 1 factors out a 6.

$$= 6 \det \begin{pmatrix} 0 & -1 & 2 \\ -1 & -1 & 1 \\ 0 & -3 & 2 \end{pmatrix}$$

combo  $(1, 3, 1)$ , combo  $(2, 1, 2)$ .

$$= -6 \det \begin{pmatrix} -1 & -1 & 1 \\ 0 & -1 & 2 \\ 0 & -3 & 2 \end{pmatrix}$$

swap  $(1, 2)$ . Swap changes the sign of the determinant.

$$= 6 \det \begin{pmatrix} 1 & 1 & -1 \\ 0 & -1 & 2 \\ 0 & -3 & 2 \end{pmatrix}$$

Multiply rule  $m = -1$  on row 1.

$$= 6 \det \begin{pmatrix} 1 & 1 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & -4 \end{pmatrix}$$

combo  $(2, 3, -3)$ .

$$= 6(1)(-1)(-4) = 24$$

Triangular rule. Formula verified.

## Elementary Matrices and the Four Rules

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The four rules can be stated in terms of elementary matrices as follows.

- Triangular** The value of  $\det(\mathbf{A})$  for either an upper triangular or a lower triangular matrix  $\mathbf{A}$  is the product of the diagonal elements:  $\det(\mathbf{A}) = a_{11}a_{22} \cdots a_{nn}$ . This is a one-arrow Sarrus' rule valid for dimension  $n$ .
- Swap** If  $\mathbf{E}$  is an elementary matrix for a swap rule, then  $\det(\mathbf{EA}) = (-1) \det(\mathbf{A})$ .
- Combination** If  $\mathbf{E}$  is an elementary matrix for a combination rule, then  $\det(\mathbf{EA}) = \det(\mathbf{A})$ .
- Multiply** If  $\mathbf{E}$  is an elementary matrix for a multiply rule with multiplier  $m \neq 0$ , then  $\det(\mathbf{EA}) = m \det(\mathbf{A})$ .

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Because  $\det(\mathbf{E}) = 1$  for a combination rule,  $\det(\mathbf{E}) = -1$  for a swap rule and  $\det(\mathbf{E}) = c$  for a multiply rule with multiplier  $c \neq 0$ , it follows that for any elementary matrix  $\mathbf{E}$  there is the **determinant multiplication rule**

$$\det(\mathbf{EA}) = \det(\mathbf{E}) \det(\mathbf{A}).$$

## Special Determinant Rules

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The results are stated for rows but also hold for columns, because  $\det(A) = \det(A^T)$ .

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Zero row	If one row of $A$ is zero, then $\det(A) = 0$ .
Duplicate rows	If two rows of $A$ are identical, then $\det(A) = 0$ .
RREF $\neq I$	If $\text{rref}(A) \neq I$ , then $\det(A) = 0$ .
Common factor	The relation $\det(A) = c \det(B)$ holds, provided $A$ and $B$ differ only in one row, say row $j$ , for which $\text{row}(A, j) = c \text{row}(B, j)$ .
Row linearity	The relation $\det(A) = \det(B) + \det(C)$ holds, provided $A$ , $B$ and $C$ differ only in one row, say row $j$ , for which $\text{row}(A, j) = \text{row}(B, j) + \text{row}(C, j)$ .

## Cofactor Expansion for $3 \times 3$ Matrices

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This is a review the college algebra topic, where the dimension of  $A$  is **3**.

**Cofactor row expansion** means the following formulas are valid:

$$\begin{aligned} |A| &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ &= a_{11}(+1) \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}(-1) \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}(+1) \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ &= a_{21}(-1) \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{22}(+1) \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + a_{23}(-1) \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \\ &= a_{31}(+1) \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} + a_{32}(-1) \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} + a_{33}(+1) \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{aligned}$$

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The formulas expand a  $3 \times 3$  determinant in terms of  $2 \times 2$  determinants, along a row of  $A$ . The attached signs  $\pm 1$  are called the **checkerboard signs**, to be defined shortly. The  $2 \times 2$  determinants are called **minors** of the  $3 \times 3$  determinant  $|A|$ . The checkerboard sign together with a minor is called a **cofactor**.

## Cofactor Expansion Illustration

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Cofactor expansion formulas are generally used when a row has one or two zeros, making it unnecessary to evaluate one or two of the  $2 \times 2$  determinants in the expansion. To illustrate, row 1 cofactor expansion gives

$$\begin{aligned} \begin{vmatrix} 3 & 0 & 0 \\ 2 & 1 & 7 \\ 5 & 4 & 8 \end{vmatrix} &= 3(+1) \begin{vmatrix} 1 & 7 \\ 4 & 8 \end{vmatrix} + 0(-1) \begin{vmatrix} 2 & 7 \\ 5 & 8 \end{vmatrix} + 0(+1) \begin{vmatrix} 2 & 1 \\ 5 & 4 \end{vmatrix} \\ &= 3(+1)(8 - 28) + 0 + 0 \\ &= -60. \end{aligned}$$

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What has been said for rows also applies to columns, due to the transpose formula

$$\det(\mathbf{A}) = \det(\mathbf{A}^T).$$

## Minor

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The  $(n - 1) \times (n - 1)$  determinant obtained from  $\det(\mathbf{A})$  by striking out row  $i$  and column  $j$  is called the  $(i, j)$ -minor of  $\mathbf{A}$  and denoted  $\mathbf{minor}(\mathbf{A}, i, j)$ . Literature might use  $M_{ij}$  for a minor.

## Cofactor

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The  $(i, j)$ -cofactor of  $\mathbf{A}$  is  $\mathbf{cof}(\mathbf{A}, i, j) = (-1)^{i+j} \mathbf{minor}(\mathbf{A}, i, j)$ .

Multiplicative factor  $(-1)^{i+j}$  is called the **checkerboard sign**, because its value can be determined by counting *plus, minus, plus*, etc., from location  $(1, 1)$  to location  $(i, j)$  in any checkerboard fashion.

## Expansion of Determinants by Cofactors

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$$(11) \quad \det(\mathbf{A}) = \sum_{j=1}^n a_{kj} \mathbf{cof}(\mathbf{A}, k, j), \quad \det(\mathbf{A}) = \sum_{i=1}^n a_{i\ell} \mathbf{cof}(\mathbf{A}, i, \ell),$$

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In (11),  $1 \leq k \leq n$ ,  $1 \leq \ell \leq n$ . The first expansion is called a **cofactor row expansion** and the second is called a **cofactor column expansion**. The value  $\mathbf{cof}(\mathbf{A}, i, j)$  is the cofactor of element  $a_{ij}$  in  $\det(\mathbf{A})$ , that is, the checkerboard sign times the minor of  $a_{ij}$ .

**2 Example (Hybrid Method)** Justify by cofactor expansion and the four properties the identity

$$\det \begin{pmatrix} 10 & 5 & 0 \\ 11 & 5 & a \\ 10 & 2 & b \end{pmatrix} = 5(6a - b).$$

**Solution:** Let  $D$  denote the value of the determinant. Then

$$D = \det \begin{pmatrix} 10 & 5 & 0 \\ 11 & 5 & a \\ 10 & 2 & b \end{pmatrix}$$

Given.

$$= \det \begin{pmatrix} 10 & 5 & 0 \\ 1 & 0 & a \\ 0 & -3 & b \end{pmatrix}$$

Combination leaves the determinant unchanged:  
combo  $(1, 2, -1)$ , combo  $(1, 3, -1)$ .

$$= \det \begin{pmatrix} 0 & 5 & -10a \\ 1 & 0 & a \\ 0 & -3 & b \end{pmatrix}$$

combo  $(2, 1, -10)$ .

$$= (1)(-1) \det \begin{pmatrix} 5 & -10a \\ -3 & b \end{pmatrix}$$

Cofactor expansion on column 1.

$$= (1)(-1)(5b - 30a)$$

Sarrus' rule for  $n = 2$ .

$$= 5(6a - b).$$

Formula verified.

**3 Example (Cramer's Rule)** Solve by Cramer's rule the system of equations

$$\begin{aligned}2x_1 + 3x_2 + x_3 - x_4 &= 1, \\x_1 + x_2 - x_4 &= -1, \\3x_2 + x_3 + x_4 &= 3, \\x_1 + x_3 - x_4 &= 0,\end{aligned}$$

verifying  $x_1 = 1, x_2 = 0, x_3 = 1, x_4 = 2$ .

**Solution:** Form the four determinants  $\Delta_1, \dots, \Delta_4$  from the base determinant  $\Delta$  as follows:

$$\Delta = \det \begin{pmatrix} 2 & 3 & 1 & -1 \\ 1 & 1 & 0 & -1 \\ 0 & 3 & 1 & 1 \\ 1 & 0 & 1 & -1 \end{pmatrix},$$

$$\Delta_1 = \det \begin{pmatrix} 1 & 3 & 1 & -1 \\ -1 & 1 & 0 & -1 \\ 3 & 3 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \quad \Delta_2 = \det \begin{pmatrix} 2 & 1 & 1 & -1 \\ 1 & -1 & 0 & -1 \\ 0 & 3 & 1 & 1 \\ 1 & 0 & 1 & -1 \end{pmatrix},$$

$$\Delta_3 = \det \begin{pmatrix} 2 & 3 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 0 & 3 & 3 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix}, \quad \Delta_4 = \det \begin{pmatrix} 2 & 3 & 1 & 1 \\ 1 & 1 & 0 & -1 \\ 0 & 3 & 1 & 3 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

Five repetitions of the methods used in the previous examples give the answers  $\Delta = -2$ ,  $\Delta_1 = -2$ ,  $\Delta_2 = 0$ ,  $\Delta_3 = -2$ ,  $\Delta_4 = -4$ , therefore Cramer's rule implies the solution

$$x_1 = \frac{\Delta_1}{\Delta}, \quad x_2 = \frac{\Delta_2}{\Delta}, \quad x_3 = \frac{\Delta_3}{\Delta}, \quad x_4 = \frac{\Delta_4}{\Delta}.$$

Then  $x_1 = 1$ ,  $x_2 = 0$ ,  $x_3 = 1$ ,  $x_4 = 2$ .

## Maple Code for Cramer's Rule

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The details of the computation above can be checked in computer algebra system maple as follows.

```
with(linalg):  
A:=matrix([  
  [2, 3, 1, -1], [1, 1, 0, -1],  
  [0, 3, 1, 1], [1, 0, 1, -1]]);  
Delta:= det(A);  
b:=vector([1,-1,3,0]):  
B1:=A: col(B1,1):=b:  
Delta1:=det(B1);  
x[1]:=Delta1/Delta;
```

## The Adjugate Matrix

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The **adjugate**  $\mathbf{adj}(A)$  of an  $n \times n$  matrix  $A$  is the transpose of the matrix of cofactors,

$$\mathbf{adj}(A) = \begin{pmatrix} \mathbf{cof}(A, 1, 1) & \mathbf{cof}(A, 1, 2) & \cdots & \mathbf{cof}(A, 1, n) \\ \mathbf{cof}(A, 2, 1) & \mathbf{cof}(A, 2, 2) & \cdots & \mathbf{cof}(A, 2, n) \\ \vdots & \vdots & \cdots & \vdots \\ \mathbf{cof}(A, n, 1) & \mathbf{cof}(A, n, 2) & \cdots & \mathbf{cof}(A, n, n) \end{pmatrix}^T.$$

A cofactor  $\mathbf{cof}(A, i, j)$  is the checkerboard sign  $(-1)^{i+j}$  times the corresponding minor determinant  $\mathbf{minor}(A, i, j)$ .

## Adjugate of a $2 \times 2$

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$$\mathbf{adj} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

In words: *swap the diagonal elements and change the sign of the off-diagonal elements.*

## Adjugate Formula for the Inverse

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For any  $n \times n$  matrix

$$\mathbf{A} \cdot \mathbf{adj}(\mathbf{A}) = \mathbf{adj}(\mathbf{A}) \cdot \mathbf{A} = \det(\mathbf{A}) \mathbf{I}.$$

The equation is valid even if  $\mathbf{A}$  is not invertible. The relation suggests several ways to find  $\det(\mathbf{A})$  from  $\mathbf{A}$  and  $\mathbf{adj}(\mathbf{A})$  with one dot product.

For an invertible matrix  $\mathbf{A}$ , the relation implies  $\mathbf{A}^{-1} = \mathbf{adj}(\mathbf{A}) / \det(\mathbf{A})$ :

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} \begin{pmatrix} \mathbf{cof}(\mathbf{A}, 1, 1) & \mathbf{cof}(\mathbf{A}, 1, 2) & \cdots & \mathbf{cof}(\mathbf{A}, 1, n) \\ \mathbf{cof}(\mathbf{A}, 2, 1) & \mathbf{cof}(\mathbf{A}, 2, 2) & \cdots & \mathbf{cof}(\mathbf{A}, 2, n) \\ \vdots & \vdots & \cdots & \vdots \\ \mathbf{cof}(\mathbf{A}, n, 1) & \mathbf{cof}(\mathbf{A}, n, 2) & \cdots & \mathbf{cof}(\mathbf{A}, n, n) \end{pmatrix}^T$$

## Elementary Matrices

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### Theorem 1 (Determinants and Elementary Matrices)

Let  $E$  be an  $n \times n$  elementary matrix. Then

Combination  $\det(E) = 1$

Multiply  $\det(E) = m$  for multiplier  $m$ .

Swap  $\det(E) = -1$

Product  $\det(EX) = \det(E) \det(X)$  for all  $n \times n$  matrices  $X$ .

### Theorem 2 (Determinants and Invertible Matrices)

Let  $A$  be a given invertible matrix. Then

$$\det(A) = \frac{(-1)^s}{m_1 m_2 \cdots m_r}$$

where  $s$  is the number of swap rules applied and  $m_1, m_2, \dots, m_r$  are the nonzero multipliers used in multiply rules when  $A$  is reduced to  $\text{rref}(A)$ .

## Determinant Products

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### Theorem 3 (Determinant Product Rule)

Let  $A$  and  $B$  be given  $n \times n$  matrices. Then

$$\det(AB) = \det(A) \det(B).$$

### Proof

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Assume one of  $A$  or  $B$  has zero determinant. Then  $\det(A) \det(B) = 0$ . If  $\det(B) = 0$ , then  $B\mathbf{x} = \mathbf{0}$  has infinitely many solutions, in particular a nonzero solution  $\mathbf{x}$ . Multiply  $B\mathbf{x} = \mathbf{0}$  by  $A$ , then  $AB\mathbf{x} = \mathbf{0}$  which implies  $AB$  is not invertible. Then the identity  $\det(AB) = \det(A) \det(B)$  holds, because both sides are zero. If  $\det(B) \neq 0$  but  $\det(A) = 0$ , then there is a nonzero  $\mathbf{y}$  with  $A\mathbf{y} = \mathbf{0}$ . Define  $\mathbf{x} = AB^{-1}\mathbf{y}$ . Then  $AB\mathbf{x} = A\mathbf{y} = \mathbf{0}$ , with  $\mathbf{x} \neq \mathbf{0}$ , which implies the identity holds.. This completes the proof when one of  $A$  or  $B$  is not invertible.

Assume  $A$ ,  $B$  are invertible and then  $C = AB$  is invertible. In particular,  $\text{rref}(A^{-1}) = \text{rref}(B^{-1}) = I$ . Write  $I = \text{rref}(A^{-1}) = E_1 E_2 \cdots E_k A^{-1}$  and  $I = \text{rref}(B^{-1}) = F_1 F_2 \cdots F_m B^{-1}$  for elementary matrices  $E_i, F_j$ . Then

$$(12) \quad AB = E_1 E_2 \cdots E_k F_1 F_2 \cdots F_m.$$

The theorem follows from repeated application of the basic identity  $\det(EX) = \det(E) \det(X)$  to relation (12), because

$$\det(A) = \det(E_1) \cdots \det(E_k), \quad \det(B) = \det(F_1) \cdots \det(F_m).$$

