Autonomous Differential Equations. The equation

$$y'(x) = f(y(x)) \tag{1}$$

has right side independent of x, hence there are no external control terms that depend on x. Due to the lack of external controls, the equation is said to be **self-governing** or **autonomous**.

A **phase line diagram** for the autonomous equation $y' = f(y)$ is a line segment with labels **sink**, **source** or **node**, one for each root of $f(y) = 0$, i.e., each equilibrium; see Figure 1. It summarizes the contents of a direction field and threaded curves, including all equilibrium solutions.

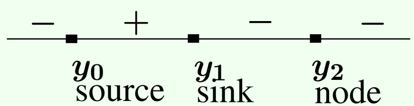

Figure 1. A phase line diagram for an autonomous equation $y' = f(y)$.

Drawing Phase Diagrams.

A phase line diagram is used to draw a **phase diagram** of threaded solutions and equilibrium solutions by using the three rules below.

1. Equilibrium solutions are horizontal lines in the phase diagram.

2. Threaded solutions of $y' = f(x, y)$ don't cross. In particular, they don't cross equilibrium solutions.

3. A threaded non-equilibrium solution that starts at $x = 0$ at a point y_0 must be increasing if $f(y_0) > 0$, and decreasing if $f(y_0) < 0$.

To justify **3**, let $y_1(x)$ be a solution with $y_1'(x) = f(y_1(x))$ either positive or negative at $x = 0$. If $y_1'(x_1) = 0$ for some $x_1 > 0$, then let $c = y_1(x_1)$ and define equilibrium solution $y_2(x) = c$. Then solution y_1 crosses an equilibrium solution at $x = x_1$, violating rule **2**.

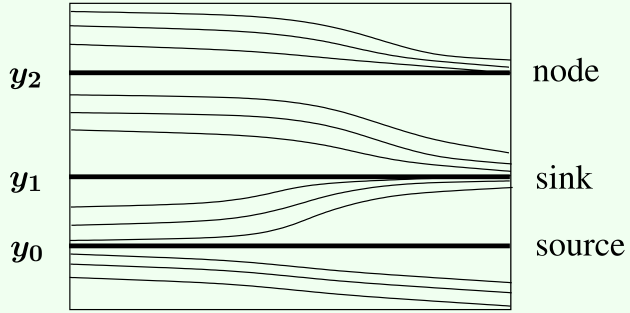

Figure 2. A phase diagram for an autonomous equation $y' = f(y)$.
The graphic is drawn directly from phase line diagram Figure 1, using rules **1**, **2**, **3**.

Direction Field Plots. A direction field for $y' = f(y)$ can be constructed in two steps. First, draw it along the y-axis. Secondly, duplicate the y-axis field at even divisions along the x-axis.

Fact 1. An equilibrium is a horizontal line. It is *stable* if all solutions starting near the line remain nearby as $x \longrightarrow \infty$.

Fact 2. Solutions don't cross. In particular, any solution that starts above or below an equilibrium solution must remain above or below.

Fact 3. A solution curve of $y' = f(y)$ rigidly moved to the left or right will remain a solution, i.e., the translate $y(x - x_0)$ of a solution to $y' = f(y)$ is also a solution.

A phase line diagram is merely a summary of the solution behavior in a direction field. Conversely, an independently made phase line diagram can be used to enrich the detail in a direction field.

Fact 3 is used to make additional threaded solutions from an initial threaded solution, by translation. Threaded solutions with turning points are observed to have their turning points march monotonically to the left, or to the right.