

Math 2250 Maple Project 5: Linear Algebra
January 2008

Due date: See the internet due dates. Maple lab 5 has problems L5.1, L5.2, L5.3.

References: Code in `maple` appears in `2250mapleL5-S2008.txt` at URL <http://www.math.utah.edu/~gustafso/>. This document: `2250mapleL5-S2008.pdf`.

Problem L5.1. (Matrix Algebra)

Define $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$. Create a worksheet in `maple` which states this problem in text, then defines the four objects. The worksheet should contain text, `maple` code and displays. Continue with this worksheet to answer (1)–(7) below. Submit problem L5.1 as a worksheet printed on 8.5 by 11 inch paper. See Example 1 for `maple` commands.

- (1) Compute AB and BA . Are they the same?
- (2) Compute $A + B$ and $B + A$. Are they the same?
- (3) Let $C = A + B$. Compare C^2 to $A^2 + 2AB + B^2$. Explain why they are different.
- (4) Compute transposes $C_1 = (AB)^T$, $C_2 = A^T$ and $C_3 = B^T$. Find an equation for C_1 in terms of C_2 and C_3 . Verify the equation.
- (5) Solve for \mathbf{X} in $B\mathbf{X} = \mathbf{v}$ by three different methods.
- (6) Solve $A\mathbf{Y} = \mathbf{v}$ for \mathbf{Y} . Do an answer check.
- (7) Solve $A\mathbf{Z} = \mathbf{w}$. Explain your answer using the three possibilities for a linear system.

Problem L5.2. (Row space)

Let $A = \begin{pmatrix} 1 & 1 & 1 & 2 & 6 \\ 2 & 3 & -2 & 1 & -3 \\ 0 & 1 & -4 & -3 & -15 \\ 1 & 2 & -3 & -1 & -9 \end{pmatrix}$. Find two different bases for the row space of A , using the following three methods.

1. The method of Example 2, below.
2. The `maple` command `rowSPACE(A)`.
3. The `rref`-method: select rows from `rref(A)`.

Two of the methods produce exactly the same basis. Verify that the two bases $\mathcal{B}_1 = \{\mathbf{v}_1, \mathbf{v}_2\}$ and $\mathcal{B}_2 = \{\mathbf{w}_1, \mathbf{w}_2\}$ are **equivalent**. This means that each vector in \mathcal{B}_1 is a linear combination of the vectors in \mathcal{B}_2 , and conversely, that each vector in \mathcal{B}_2 is a linear combination of the vectors in \mathcal{B}_1 .

Problem L5.3. (Matrix Equations)

Let $A = \begin{pmatrix} 8 & 10 & 3 \\ -3 & -5 & -3 \\ -4 & -4 & 1 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$. Let P denote a 3×3 matrix. Assume the following result:

Lemma 1. The equality $AP = PT$ holds if and only if the columns \mathbf{v}_1 , \mathbf{v}_2 , \mathbf{v}_3 of P satisfy $A\mathbf{v}_1 = \mathbf{v}_1$, $A\mathbf{v}_2 = -2\mathbf{v}_2$, $A\mathbf{v}_3 = 5\mathbf{v}_3$. [proved after Example 4]

- (a) Determine three specific columns for P such that $\det(P) \neq 0$ and $AP = PT$. Infinitely many answers are possible. See Example 4 for the `maple` method that determines a column of P .
- (b) After reporting the three columns, check the answer by computing $AP - PT$ (it should be zero) and $\det(P)$ (it should be nonzero).

Example 1. Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 0 & -1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 9 \\ 8 \\ 3 \end{pmatrix}$. Create a **maple** work sheet. Define and display matrix A and vector \mathbf{b} . Then compute

- (1) The inverse of A .
- (2) The augmented matrix $C = \mathbf{aug}(A, \mathbf{b})$.
- (3) The reduced row echelon form $R = \mathbf{rref}(C)$.
- (4) The column \mathbf{X} of R which solves $A\mathbf{X} = \mathbf{b}$.
- (5) The matrix A^3 .
- (6) The transpose of A .
- (7) The matrix $A - 3A^2$.
- (8) The solution \mathbf{X} of $A\mathbf{X} = \mathbf{b}$ by two methods different than (4).

Solution: A lab instructor can help you to create a blank work sheet in **maple**, enter code and print the work sheet. The code to be entered appears below. To get help, enter `?linalg` into a worksheet, then select commands that match ones below.

```
with(linalg):
A:=matrix([[1,2,3],[2,-1,1],[3,0,-1]]);
b:=vector([9,8,3]);
print("(1)"); inverse(A);
print("(2)"); C:=augment(A,b);
print("(3)"); R:=rref(C);
print("(4)"); X:=col(R,4);
print("(5)"); evalm(A^3);
print("(6)"); transpose(A);
print("(7)"); evalm(A-3*(A^2));
print("(8)"); X:=linsolve(A,b); X:=evalm(inverse(A) &* b);
```

Example 2. Let $A = \begin{pmatrix} 1 & 1 & 1 & 2 & 6 \\ 2 & 3 & -2 & 1 & -3 \\ 3 & 5 & -5 & 1 & -8 \\ 4 & 3 & 8 & 2 & 3 \end{pmatrix}$.

- (1) Find a basis for the column space of A .
- (2) Find a basis for the row space of A .
- (3) Find a basis for the nullspace of A .
- (4) Find $\mathbf{rank}(A)$ and $\mathbf{nullity}(A)$.
- (5) Find the dimensions of the nullspace, row space and column space of A .

Solution: The theory applied: *The columns of B corresponding to the leading ones in $\mathbf{rref}(B)$ are independent and form a basis for the column space of B .* These columns are called the **pivot columns** of B . Results for the row space can be obtained by applying the above theory to the transpose of the matrix.

The **maple** code which applies is

```
with(linalg):
A:=matrix([[ 1, 1, 1, 2, 6],
           [ 2, 3,-2, 1,-3],
           [ 3, 5,-5, 1,-8],
           [ 4, 3, 8, 2, 3]]);
print("(1)"); C:=rref(A); # leading ones in columns 1,2,4
           BASIScolumnspace=col(A,1),col(A,2),col(A,4);
```

```

print("(2)"); F:=rref(transpose(A)); # leading ones in columns 1,2,3
      BASISrowSpace=row(A,1),row(A,2),row(A,3);
print("(3)"); nullspace(A); linsolve(A,vector([0,0,0,0]));
print("(4)"); RANK=rank(A); NULLITY=coldim(A)-rank(A);
print("(5)"); DIMnullspace=coldim(A)-rank(A); DIMrowSpace=rank(A);
      DIMcolumnSpace=rank(A);

```

Example 3. Let $A = \begin{pmatrix} 1 & 1 & 1 & 2 & 6 \\ 2 & 3 & -2 & 1 & -3 \\ 3 & 5 & -5 & 1 & -8 \\ 4 & 3 & 8 & 2 & 3 \end{pmatrix}$. Verify that the following column space bases of A are equivalent.

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 3 \\ 5 \\ 3 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 2 \end{pmatrix},$$

$$\mathbf{w}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -3 \end{pmatrix}, \quad \mathbf{w}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 17 \end{pmatrix}, \quad \mathbf{w}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -9 \end{pmatrix}.$$

Solution: We will use this result:

Lemma 2. Bases $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ are equivalent bases if and only if the augmented matrices $F = \text{aug}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$, $G = \text{aug}(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)$ and $H = \text{aug}(F, G)$ satisfy the rank condition $\text{rank}(F) = \text{rank}(G) = \text{rank}(H) = 3$.

The proof appears below.

The maple code which applies is

```

with(linalg):
A:=matrix([[ 1, 1, 1, 2, 6],
           [ 2, 3,-2, 1,-3],
           [ 3, 5,-5, 1,-8],
           [ 4, 3, 8, 2, 3]]);
v1:=vector([1,2,3,4]); v2:=vector([1,3,5,3]); v3:=vector([2,1,1,2]);
w1:=vector([1, 0, 0, -3]); w2:=vector([0, 1, 0, 17]); w3:=vector([0, 0, 1, -9]);
F:=augment(v1,v2,v3);
G:=augment(w1,w2,w3);
H:=augment(F,G);
rank(F); rank(G); rank(H);

```

We remark that the two bases in the example were discovered from the maple code

```

rref(A); # pivot cols 1,2,4
v1:=col(A,1); v2:=col(A,2); v3:=col(A,4);
B:=rref(transpose(A)); # pivot cols 1,2,3
w1:=row(B,1); w2:=row(B,2); w3:=row(B,3);

```

Proof of Lemma 2.

Proof: The test appears in the online pdf documents at the course web site. Let's justify the test here, independently, showing only half the proof: $\text{rank}(F) = \text{rank}(G) = \text{rank}(H) = n$ implies the bases are equivalent.

The equation $\mathbf{rref}(F) = EF$ holds for E a product of elementary matrices. Then EH has to have n lead variables, because of F in the first n columns, and the remaining rows are zero, because $\text{rank}(H) = n$. Therefore, the first n columns of H are the pivot columns of H . The non-pivots of H are just the columns of G , and by the pivot theory, they are linear combinations of the pivot columns, namely, the columns of F . We can multiply H by a permutation matrix P which effectively swaps F and G . Already, HP has the n independent columns of F , so its rank is at least n . But its other columns are linear combinations of these columns, so the rank is exactly n . Now we argue by symmetry that the columns of F are linear combinations of the columns of G , using HP instead of H .

The proof is complete.

Example 4. Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 0 & 0 \end{pmatrix}$. Solve the equation $A\mathbf{x} = -3\mathbf{x}$ for \mathbf{x} .

Solution. Let $\lambda = -3$. The idea is to write the equation $A\mathbf{x} = \lambda\mathbf{x}$ as a homogeneous problem $(A - \lambda I)\mathbf{x} = \mathbf{0}$. Define $B = A - \lambda I$. The homogeneous equation $B\mathbf{x} = \mathbf{0}$ always has the solution $\mathbf{x} = \mathbf{0}$. It has a nonzero solution \mathbf{x} if and only if there are infinitely many solutions, in which case the solutions are found by a frame sequence to $\mathbf{rref}(B)$. The `maple` details appear below. The basis vectors for $B\mathbf{x} = \mathbf{0}$ are obtained in the usual way, by taking partial derivatives on the general solution with respect to the symbols t_1, t_2, \dots . In this case, there is just one basis vector

$$\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}.$$

```
with(linalg):
A:=matrix([[1,2,3],[2,-1,1],[3,0,0]]);
B:=evalm(A-(-3)*diag(1,1,1));
linsolve(B,vector([0,0,0]));
# ans: t_1*vector([-2,1,2])
# Basis == partial on t_1 == vector([-2,1,2])
```

Proof of Lemma 1. Define $r_1 = 1, r_2 = -2, r_3 = 5$. Assume $AP = PT, P = \mathbf{aug}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ and $T = \mathbf{diag}(r_1, r_2, r_3)$. The definition of matrix multiplication implies that $AP = \mathbf{aug}(A\mathbf{v}_1, A\mathbf{v}_2, A\mathbf{v}_3)$ and $PT = \mathbf{aug}(r_1\mathbf{v}_1, r_2\mathbf{v}_2, r_3\mathbf{v}_3)$. Then $AP = PT$ holds if and only if the columns of the two matrices match, which is equivalent to the three equations $A\mathbf{v}_1 = r_1\mathbf{v}_1, A\mathbf{v}_2 = r_2\mathbf{v}_2, A\mathbf{v}_3 = r_3\mathbf{v}_3$. The proof is complete.

End of Maple Lab 5 Linear Algebra.