Differential Equations and Linear Algebra

2250-1 7:30am 28 April 2008

Instructions. The time allowed is 120 minutes. The examination consists of six problems, one for each of chapters 3, 4, 5, 6, 7, 10, each problem with multiple parts. A chapter represents 20 minutes on the final exam.

Each problem on the final exam represents several textbook problems numbered (a), (b), (c), $\cdots$. Choose the problems to be graded by check-mark $\boxed{\text{X}}$; the credits should add to 100. Each chapter (Ch3, Ch4, Ch5, Ch6, Ch7, Ch10) adds at most 100 towards the maximum final exam score of 600. The final exam score is reported as a percentage 0 to 100, which is the sum of the scores earned on six chapters divided by 600 to make a fraction, then converted to a percentage.

- Calculators, books, notes and computers are not allowed.
- Details count. Less than full credit is earned for an answer only, when details were expected. Generally, answers count only 25% towards the problem credit.
- Completely blank pages count 40% or less, at the whim of the grader.
- Answer checks are not expected or required. First drafts are expected, not complete presentations.
- Please submit **exactly six** separately stapled packages of problems, one package per chapter.

Final Grade. The final exam counts as two midterm exams. For example, if exam scores earned were 90, 91, 92 and the final exam score is 89, then the exam average for the course is

$$\text{Exam Average} = \frac{90 + 91 + 92 + 89 + 89}{5} = 90.2.$$

Dailies count 30% of the final grade. The course average is computed from the formula

$$\text{Course Average} = \frac{70}{100}(\text{Exam Average}) + \frac{30}{100}(\text{Dailies Average}).$$

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Ch3. (Linear Systems and Matrices)

☐ [40%] Ch3(a): Let B be the invertible matrix given below, where $\boxed{?}$ means the value of the entry does not affect the answer to this problem. The second matrix C is the adjugate (or adjoint) of B. Let A be a matrix such that $BC^T(A^2C^3 + B^2CA^T) = 0$. Find all possible values of $\det(A)$.
Notation: X^T is the transpose of X. And X^2 means XX.

$$B = \begin{pmatrix} 1 & -1 & -1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 1 & 0 & 0 & 4 \end{pmatrix}, \quad C = \begin{pmatrix} 8 & ? & 4 & 0 \\ ? & ? & -4 & 0 \\ -4 & ? & 8 & ? \\ ? & -3 & ? & ? \end{pmatrix}$$

ans: $\det(A) = \frac{1}{2^{04}}$

☐ [40%] Ch3(b): State the three possibilities for a linear system $A\mathbf{x} = \mathbf{b}$ [5%]. Determine which values of k correspond to these three possibilities, for the system $A\mathbf{x} = \mathbf{b}$ given in the display below [35%].

$$A = \begin{pmatrix} 4 & 3 & -k \\ 0 & k-2 & k-4 \\ 4 & 3 & -4 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 2k-6 \\ k-3 \end{pmatrix}$$

ans: No Sol never
∞ – Many $(k-2)(k-4)=0$
Unique $(k-2)(k-4)\neq 0$

☐ [20%] Ch3(c): Assume A is an $n \times n$ matrix and that $A\mathbf{x} = \mathbf{b}$ has a solution for any nonzero vector $\mathbf{b}$. Find a basis for the set S of all vectors of the form $A\mathbf{x}$, where $\mathbf{x}$ is any vector in $\mathcal{R}^n$.

ans: Basis = all columns of I

If you solved (a), (b) and (c), then go on to Ch4. **Otherwise**, try (d), (e) and (f). Maximum credit is 100%.

☐ [10%] Ch3(d): Give an example of a 3×3 matrix A such that the system $A\mathbf{x} = \mathbf{0}$ has a solution $\mathbf{x} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$.

ans: $A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. There are ∞ – many answers A.

☐ [20%] Ch3(e): Find the value of x_3 by Cramer's Rule in the system $C\mathbf{x} = \mathbf{b}$, given C and $\mathbf{b}$ below. Evaluate determinants by any hybrid method (triangular, swap, combo, multiply, cofactor). The use of 2×2 Sarrus' rule is allowed. The 3×3 Sarrus' rule is **disallowed**.

$$C = \begin{pmatrix} 0 & 2 & -1 & 0 \\ 0 & 0 & 4 & 0 \\ 1 & 3 & -2 & 1 \\ 0 & 0 & 2 & 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}$$

ans: $x_3 = \frac{1}{4}$
$\Delta = 8$
$\Delta_3 = 2$

☐ [10%] Ch3(f): Prove or display a counterexample: the product AB of two triangular 2×2 matrices A and B is triangular. False. $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ is not triangular

Staple this page to the top of all Ch3 work. Submit one package per chapter.

Ch 3 S 2008 7:30 28 apr

Ch3 (a) $BC = \det(B) I \Rightarrow \text{row}(B,3)\,\text{col}(C,3) = \det(B) \Rightarrow \det(B) = 20$

Then $\det B \det C = (\det B)^4 \Rightarrow \det(C) = 20^3$. BC^T is invertible $\Rightarrow$

$A^2C^3 + B^2 C A^T = 0 \Rightarrow (\det A)^2 \det(C)^3 = \det(-I)\det(B)^2 \det(C) \det(A^T)$

$\Rightarrow \det A = \det(-I)\det(B)^2 / \det(C)^2 = (-1)^4\, 20^2/20^6 = \boxed{\frac{1}{20^4}}$

Ch3 (b) $C = \text{aug}(A,b) = \left(\begin{array}{ccc|c} 4 & 3 & -k & 1 \\ 0 & k-2 & k-4 & 2k-6 \\ 4 & 3 & -4 & k-3 \end{array}\right) \cong \left(\begin{array}{ccc|c} 4 & 3 & -k & 1 \\ 0 & k-2 & k-4 & 2k-6 \\ 0 & 0 & k-4 & k-4 \end{array}\right)$

$\cong \left(\begin{array}{ccc|c} 4 & 3 & -k & 1 \\ 0 & k-2 & 0 & k-2 \\ 0 & 0 & k-4 & k-4 \end{array}\right)$

- NO Sol never happens
- ∞ - many solutions if $(k-2)(k-4) = 0$
- unique sol if $(k-2)(k-4) \neq 0$

Ch3 (c) Choose $\vec{b} = \text{col}(I, i)$. Then $A\vec{x} = \vec{b}$ has a sol $\vec{x}$, which implies $\vec{b}$ is in S. Then S contains all columns of I, and they form a basis for S.

Ch3 (d) $A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ works

Ch3 (e) $x_3 = \dfrac{\Delta_3}{\Delta}$ $\quad \Delta = \begin{vmatrix} 0 & 2 & -1 & 0 \\ 0 & 0 & 4 & 0 \\ 1 & 3 & -2 & 1 \\ 0 & 0 & 2 & 1 \end{vmatrix} = (-1)(4)\begin{vmatrix} 0 & 2 & 0 \\ 1 & 3 & 1 \\ 0 & 0 & 1 \end{vmatrix} = (-1)(4)(1)(-2) = 8$

$\Delta_3 = \begin{vmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 3 & 2 & 0 \\ 0 & 0 & 3 & 1 \end{vmatrix} = (-1)(1)\begin{vmatrix} 0 & 2 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 1 \end{vmatrix} = (-1)(1)(1)(-2) = 2$

$x_3 = \dfrac{1}{4}$

Ch3 (f) $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ not triangular

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Ch4. (Vector Spaces)

☐ [25%] Ch4(a): Independence of 4 fixed vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$ can be decided by counting the pivot columns of their augmented matrix. State a different test which can decide upon independence of four vectors [10%]. Apply one of these two tests and report all values of x for which the four vectors are **dependent** [15%]. Test: Let $A = \text{aug}(v_1, v_2, v_3, v_4)$. Independent $\iff \det A \neq 0$. Applies only in case the vectors are in $\mathbb{R}^4$.

$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 2 \\ 5 \\ 3 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 0 \\ 2 \\ x \\ 3 \end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix} -2 \\ 2 \\ 9 \\ x \end{pmatrix}.$$

ans: Dependent $\iff (x-5)(x-6) = 0$

☐ [25%] Ch4(b): Consider the four vectors

$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}, \quad \mathbf{w}_1 = \begin{pmatrix} 1 \\ 2 \\ x \end{pmatrix}, \quad \mathbf{w}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

Find all values of x such that the subspaces $S_1 = \mathbf{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ and $S_2 = \mathbf{span}\{\mathbf{w}_1, \mathbf{w}_2\}$ are equal, showing all details. Ans: $x + 3 = 0$

☐ [50%] Ch4(c): Define the 5×5 matrix A by the display below. Find a basis of fixed vectors in $\mathcal{R}^5$ for (1) the column space of A [25%] and (2) the row space of A [25%]. The two displayed bases **must** consist of columns of A and columns of A^T (the transpose of A), respectively.

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & -3 & 0 & 1 \\ 0 & -1 & -2 & 0 & 1 \\ 0 & 6 & 6 & 0 & 0 \\ 0 & 2 & 4 & 0 & 0 \end{pmatrix}$$

ans: Colspace = span cols 2, 3 of A

rowspace = span cols 2, 3, 4 of A^T

If you finished (a), (b) and (c), then 100% has been marked – go on to Ch5. **Otherwise**, unmark one or more of (a), (b) or (c), then complete (d) and/or (e). A maximum of four problems will be graded.

☐ [25%] Ch4(d): Define S to be the set of all vectors $\mathbf{x}$ in $\mathcal{R}^4$ which are orthogonal to all the vectors $\mathbf{y}$ satisfying $y_1 + y_2 = 0$, $y_1 + y_3 + 2y_4 = 0$. Prove or disprove that S is a subspace of $\mathcal{R}^4$.

☐ [25%] Ch4(e): Find a 4×4 system of linear equations for the constants a, b, c, d in the partial fractions decomposition below [10%]. Solve for a, b, c, d, showing all solution steps [10%]. Report the answers [5%].

$$\frac{8x^3 + 8x^2 - 12x + 12}{(x-1)^2(x+1)^2} = \frac{a}{x-1} + \frac{b}{(x-1)^2} + \frac{c}{x+1} + \frac{d}{(x+1)^2}$$

ans: $a=3$, $b=4$, $c=5$, $d=6$

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Ch4 (a)

$$\det(A) = \begin{vmatrix} 2 & 0 & 0 & -2 \\ 2 & 2 & 2 & 2 \\ 1 & 5 & x & 9 \\ 0 & 3 & 3 & x \end{vmatrix} = 2\begin{vmatrix} 2 & 2 & 2 \\ 5 & x & 9 \\ 3 & 3 & x \end{vmatrix} + (-1)(-2)\begin{vmatrix} 2 & 2 & 2 \\ 1 & 5 & x \\ 0 & 3 & 3 \end{vmatrix}$$

← cofactor rule

← multiply rule 3X

$$= 4\begin{vmatrix} 1 & 1 & 1 \\ 5 & x & 9 \\ 3 & 3 & x \end{vmatrix} + 12\begin{vmatrix} 1 & 1 & 1 \\ 1 & 5 & x \\ 0 & 1 & 1 \end{vmatrix}$$

$$= 4\begin{vmatrix} 1 & 1 & 1 \\ 0 & x-5 & 4 \\ 0 & 0 & x-3 \end{vmatrix} + 12\begin{vmatrix} 1 & 0 & 0 \\ 1 & 5 & x \\ 0 & 1 & 1 \end{vmatrix}$$ ← combo rule

$$= 4(x-5)(x-3) + 12(5-x)$$
$$= 4[\,x^2 - 8x + 15 + 15 - 3x\,]$$
$$= 4[\,x^2 - 11x + 30\,] = 4[\,x-5\,][x-6]$$

Ch4 (b) $\{v_1, v_2\}$ is independent because v_1 is not a multiple of v_2.
$\{w_1, w_2\}$ is independent for the same reason.
Let $A = \text{aug}(v_1, v_2)$, $B = \text{aug}(w_1, w_2)$, $C = \text{aug}(A, B)$.

$$C = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 2 & -1 \\ -3 & -3 & x & 0 \end{pmatrix} \cong \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 2 & -1 \\ 0 & 0 & x+3 & 0 \end{pmatrix} \cong \begin{pmatrix} 0 & -3 & -3 & 3 \\ 1 & 2 & 2 & -1 \\ 0 & 0 & x+3 & 0 \end{pmatrix}$$

<u>Thm</u>: $S_1 = S_2 \iff \text{rank } A = \text{rank } B = \text{rank } C = 2$

ans: $S_1 = S_2 \iff x + 3 = 0$

Ch4 (c)

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & -3 & 0 & 1 \\ 0 & -1 & -2 & 0 & 1 \\ 0 & 6 & 6 & 0 & 0 \\ 0 & 2 & 4 & 0 & 0 \end{pmatrix} \cong \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & -1 & -2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \cong \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

pivots of $A = 2, 3$

$$A^T = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & -1 & 6 & 2 \\ 0 & -3 & -2 & 6 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix} \cong \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & -3 & -1 & 6 & 2 \\ 0 & -3 & -2 & 6 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \cong \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 6 & 2 \\ 0 & 0 & 1 & 6 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

pivots of $A^T = 2, 3, 4$

Ch4 (d) write $B\vec{y} = 0$ where $B = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 2 \end{pmatrix}$. (1) Zero $\vec{0}$ is in S because $\vec{0} \cdot \vec{y} = 0$ for all $\vec{y}$. (2) Let $\vec{X}_1, \vec{X}_2$ be in S and define $\vec{X} = c_1\vec{X}_1 + c_2\vec{X}_2$. Then $\vec{X} \cdot \vec{y} = c_1 \vec{X}_1 \cdot \vec{y} + c_2 \vec{X}_2 \cdot \vec{y} = c_1(0) + c_2(0) = 0$ for all $\vec{y}$ s.t. $B\vec{y} = \vec{0}$. By the Subspace Criterion, S is a subspace.

Ch4 (e)

$x=1$: $16 = 0 + 4b + 0 + 0$ → $b = 4$
$x=-1$: $24 = 0 + 0 + 0 + 4d$ → $d = 6$
$x=0$: $12 = -a + b + c + d$
$x=2$: $84 = 9a + 9b + 3c + d$ → $a = 3,\ c = 5$

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Ch5. (Linear Differential Equations of Higher Order)

☐ [10%] Ch5(a): Find the general solution $y(x)$ of the differential equation

$$4y'' + 21y' + 5y = 0.$$

ans: $y = c_1 e^{-5x} + c_2 e^{-x/4}$

☐ [10%] Ch5(b): Given a damped spring-mass system $mx''(t) + cx'(t) + kx(t) = 0$ with $m = 4$, $c = 21$ and $k = 5$, classify the answer $x(t)$ as over-damped, critically damped or under-damped. Please, **do not solve** the differential equation!

ans: over-damped

☐ [20%] Ch5(c): Find the general solution of the homogeneous differential equation, given it has characteristic equation

$$r^3(r^2 + 2r + 10)^2(r^2 + 2) = 0.$$

ans: linear combination of $1, x, x^2, e^{-x}\cos 3x, e^{-x}\sin 3x, xe^{-x}\cos 3x, xe^{-x}\sin 3x, \cos\sqrt{2}x, \sin\sqrt{2}x$

☐ [10%] Ch5(d): Find a fifth order linear homogeneous differential equation with constant coefficients whose general solution is $y = c_1 + (c_2 + c_3 x)e^{-x} + c_4 \cos x + c_5 \sin x$.

ans: $y^{v} + 2y^{iv} + 2y''' + 2y'' + y' = 0$

☐ [20%] Ch5(e): Find the steady-state periodic solution for the differential equation

$$x'' + 4x' + 15x = 50\cos(5t).$$

ans: $x_{ss} = 2\sin 5t - \cos 5t$

☐ [30%] Ch5(f): Assume a ninth order constant-coefficient linear differential equation has characteristic equation $r^3(r^2 + 4)(r^2 + r)^2 = 0$. Suppose the right side of the differential equation is

$$f(x) = x^2(x + 2e^{-x}) + 5x\sin 2x + \cos x$$

Determine the **corrected** trial solution for y_p according to the method of undetermined coefficients. **Do not evaluate** the undetermined coefficients!

Staple this page to the top of all Ch5 work. Submit one package per chapter.

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ch5 ⓐ $4r^2+21r+5=0 \Rightarrow (r+5)(4r+1)=0 \Rightarrow y=c_1e^{-5x}+c_2e^{-x/4}$

ch5 ⓑ over-damped by ch5 ⓐ

ch5 ⓒ $r=0,0,0,-1\pm 3i, -1\pm 3i, \pm\sqrt{2}i$

$y=c_1+c_2x+c_3x^2+e^{-x}(c_4\cos 3x+c_5\sin 3x+c_6x\cos 3x+c_7x\sin 3x)$
$+c_8\cos(\sqrt{2}x)+c_9\sin(\sqrt{2}x)$

ch5 ⓓ roots $=0,-1,-1,\pm i \Rightarrow r(r+1)^2(r^2+1)=0$

$\Rightarrow r(r^2+2r+1)(r^2+1)=0$

$\Rightarrow r(r^4+2r^3+r^2+r^2+2r+1)=0$

$\Rightarrow r^5+2r^4+2r^3+2r^2+r=0$

ans: $y^{v}+2y^{iv}+2y'''+2y''+y'=0$

ch5 ⓔ $x=d_1\cos 5t+d_2\sin 5t$

$x'=-5d_1\sin 5t+5d_2\cos 5t$

$x''=-25x$

$(-25x)+4x'+15x=50\cos 5t$

$-20d_1\sin 5t+20d_2\cos 5t-10d_1\cos 5t-10d_2\sin 5t=50\cos 5t$

$\begin{cases}-10d_1+20d_2=50\\-20d_1-10d_2=0\end{cases} \rightarrow \begin{cases}-d_1+2d_2=5\\-2d_1-d_2=0\end{cases}$ $d_1=\dfrac{\begin{vmatrix}5&2\\0&-1\end{vmatrix}}{\begin{vmatrix}-1&2\\-2&-1\end{vmatrix}}=\dfrac{-5}{5}=-1$

$d_2=-2d_1=2$

ch6 ⓔ $y=x^{s_1}(d_1+d_2x+d_3x^2+d_4x^3)$

$+x^{s_2}(d_5+d_6x+d_7x^2)e^{-x}$

$+x^{s_3}(d_8\cos 2x+d_9\sin 2x+d_{10}x\cos 2x+d_{11}x\sin 2x)$

$+x^{s_4}(d_{12}\cos x+d_{13}\sin x)$

Char eq roots $=0,0,0,0,0, \pm 2i, -1,-1$

$s_1=5$
$s_2=2$
$s_3=1$
$s_4=0$

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Ch6. (Eigenvalues and Eigenvectors)

☐ [30%] Ch6(a): Find the eigenvalues of the matrix $A = \begin{pmatrix} 0 & 2 & -1 & 0 & 0 \\ -2 & 0 & -2 & 3 & 0 \\ 0 & 0 & 5 & -1 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 7 & 1 & 3 \end{pmatrix}$.

ans: $3, 4, 4, 2i, -2i$

To save time, **do not** find eigenvectors!

☐ [20%] Ch6(b): Let A be a 3×3 matrix satisfying Fourier's model

$$A\left(c_1\begin{pmatrix}1\\-2\\0\end{pmatrix} + c_2\begin{pmatrix}-1\\3\\0\end{pmatrix} + c_3\begin{pmatrix}1\\1\\1\end{pmatrix}\right) = 4c_3\begin{pmatrix}1\\1\\1\end{pmatrix}.$$

Prove or disprove: $\mathbf{x} = \begin{pmatrix}1\\2\\0\end{pmatrix}$ is an eigenvector of A.

ans: $\left(0, \begin{pmatrix}1\\2\\0\end{pmatrix}\right)$ is an eigenpair

☐ [15%] Ch6(c): Find a 2×2 matrix A with eigenpairs

$$\left(2, \begin{pmatrix}1\\2\end{pmatrix}\right), \quad \left(-2, \begin{pmatrix}1\\4\end{pmatrix}\right).$$

ans: $A = \begin{pmatrix} 6 & -2 \\ 16 & -6 \end{pmatrix}$

☐ [35%] Ch6(d): The matrix A below has eigenvalues 3, 3 and 3. Test A to see it is diagonalizable, and if it is, then display Fourier's model for A.

$$A = \begin{pmatrix} 4 & 1 & 1 \\ -1 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$

ans: Only one eigenpair $\left(3, \begin{pmatrix}-1\\1\\0\end{pmatrix}\right)$

not diagonalizable

☐ [20%] Ch6(e): If you were unable to earn 100% from problems (a) through (d), then solve this one, **otherwise proceed to Ch7**.

Assume A is a given 4×4 matrix with eigenvalues 0, 1, $3 \pm 2i$. Find the eigenvalues of $4A - 3I$, where I is the identity matrix.

ans: $-\frac{3}{4}, 0, \frac{3}{2} \pm \frac{3}{2}i$

Staple this page to the top of all Ch6 work. Submit one package per chapter.

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Ch6 (a) $\det(A-\lambda I) = (-\lambda)\begin{vmatrix} -\lambda & -2 & 3 & 0 \\ 0 & 5-\lambda & -1 & 0 \\ 0 & 1 & 3-\lambda & 0 \\ 0 & 7 & 1 & 3-\lambda \end{vmatrix} + (-1)(-2)\begin{vmatrix} 2 & -1 & 0 & 0 \\ 0 & 5-\lambda & -1 & 0 \\ 0 & 1 & 3-\lambda & 0 \\ 0 & 7 & 1 & 3-\lambda \end{vmatrix}$

$= (-\lambda)(-\lambda)\,\Delta + (-1)(-2)(2)\,\Delta\,, \quad \Delta = \begin{vmatrix} 5-\lambda & -1 & 0 \\ 1 & 3-\lambda & 0 \\ 7 & 1 & 3-\lambda \end{vmatrix}$

$\Delta = (3-\lambda)\begin{vmatrix} 5-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} = (3-\lambda)(\lambda^2 - 8\lambda + 16)$

$\det(A-\lambda I) = (\lambda^2+4)\,\Delta = (\lambda^2+4)(3-\lambda)(\lambda-4)^2$

Eigenvalues $= \pm 2i, 3, 4, 4$

Ch6 (b) $\begin{pmatrix} 1 & -1 & 1 \\ -2 & 3 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ has rank $= 2$ with pivots $= 1, 2 \implies$

$\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} =$ linear combination of cols 1, 2. These are eigenvectors for eigenvalue 0. So $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} =$ an eigenvector for $\lambda = 0$.

Ch6 (c) $AP = PD \quad P = \begin{pmatrix} 1 & 1 \\ 2 & 4 \end{pmatrix},\; P^{-1} = \begin{pmatrix} 4 & -1 \\ -2 & 1 \end{pmatrix}\frac{1}{2},\; D = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$

$A = PDP^{-1} = \begin{pmatrix} 1 & 1 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}P^{-1} = \begin{pmatrix} 2 & -2 \\ 4 & -8 \end{pmatrix}\frac{1}{2}\begin{pmatrix} 4 & -1 \\ -2 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & -1 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} 4 & -1 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 6 & -2 \\ 16 & -6 \end{pmatrix}$

Ch6 (d) $\det(A - \lambda I) = (3-\lambda)(\lambda^2 - 6\lambda + 9) = (3-\lambda)(\lambda-3)^2$

$A - 3I = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \cong \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ one free var. Not enough eigenvectors not diagonalizable.

Ch6 (e) $B = 4A - 3I$

$B - \lambda I = 4A - 3I - \lambda I$

$= 4\left(A - \left(\frac{3}{4} + \lambda\right) I\right)$

$\det(B - \lambda I) = 0 = 4^4 \det\left(A - \left(\frac{3}{4}+\lambda\right)I\right)$

$\frac{3}{4} + \lambda = 0, 1, 3 \pm 2i$

$3 + 4\lambda = 0, 3, 9 \pm 6i$

$4\lambda = -3, 0, 6 \pm 6i$

$\lambda = -\frac{3}{4}, 0, \frac{3}{2} \pm \frac{3}{2} i$

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Ch7. (Linear Systems of Differential Equations)

☐ [50%] Ch7(a): Apply the eigenanalysis method to solve the differential system $\mathbf{u}' = A\mathbf{u}$, given

$$A = \begin{pmatrix} -3 & 4 & -10 \\ 0 & 2 & 0 \\ 5 & -4 & 12 \end{pmatrix}$$

ans:

$$\vec{u} = c_1 e^{2t}\begin{pmatrix} 4/5 \\ 1 \\ 0 \end{pmatrix} + c_2 e^{2t}\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} + c_3 e^{7t}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

☐ [25%] Ch7(c): Solve for the general solution $x(t)$, $y(t)$ in the system below. Use any method that applies, from the four possible methods.

$$\frac{dx}{dt} = x + 3y,$$
$$\frac{dy}{dt} = 18x + 4y.$$

ans:

$$\begin{cases} x = c_1 e^{-5t} + c_2 e^{10t} \\ y = -2c_1 e^{-5t} + 3c_2 e^{10t} \end{cases}$$

☐ [15%] Ch7(c): Solve the 3×3 differential system $\mathbf{u}' = A\mathbf{u}$ for matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

ans

$$\begin{cases} x = c_1 t^2 + c_2 t + c_3 \\ y = 2c_1 t + c_2 \\ z = c_3 \end{cases}$$

☐ [10%] Ch7(d): Assume A is 3×3 and the general solution of $\mathbf{u}' = A\mathbf{u}$ is given by

$$\mathbf{u}(t) = c_1 e^t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} + c_3 e^{-t} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}.$$

Display Fourier's model for A. ans → $A\left(c_1\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} + c_3\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}\right) = c_1\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - c_3\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$

If you solved (a), (b), (c), (d), then you have marked 100%. If so, then go on to Ch10, otherwise, continue here. You may select either (d) or (e) but not both. Only 4 parts will be graded.

☐ [10%] Ch7(e): A 3×3 real matrix A has all eigenvalues equal to 0 and corresponding eigenvectors

$$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 5 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}.$$

Display the general solution of the differential equation $\mathbf{x}' = A\mathbf{x}$.

ans: $\vec{x}(t) = c_1\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ 5 \\ 1 \end{pmatrix} + c_3\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$

Staple this page to the top of all Ch7 work. Submit one package per chapter.

ch7 S2008 28apr

ch7(a) $\det(A-\lambda I) = (2-\lambda)\begin{vmatrix} -3-\lambda & -10 \\ 5 & 12-\lambda \end{vmatrix} = (2-\lambda)(\lambda^2-9\lambda-36+50)$

$= (2-\lambda)(\lambda-2)(\lambda-7)$

$B = A-2I = \begin{pmatrix} -5 & 4 & -10 \\ 0 & 0 & 0 \\ 5 & -4 & 10 \end{pmatrix} \cong \begin{pmatrix} 5 & -4 & 10 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad x_1 = \frac{4t_1}{5} - 2t_2$, $x_2 = t_1$, $x_3 = t_2$

$v_1 = \begin{pmatrix} 4/5 \\ 1 \\ 0 \end{pmatrix}$ $\quad v_2 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$

$B = A-7I = \begin{pmatrix} -10 & 4 & -10 \\ 0 & -5 & 0 \\ 5 & -4 & 5 \end{pmatrix} \cong \begin{pmatrix} -10 & 0 & -10 \\ 0 & 1 & 0 \\ 5 & 0 & 5 \end{pmatrix} \cong \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$v_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ $\quad \vec{u}(t) = c_1 e^{2t}\begin{pmatrix} 4/5 \\ 1 \\ 0 \end{pmatrix} + c_2 e^{2t}\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} + c_3 e^{7t}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

ch7(b) $r^2 - 5r - 50 = 0 \quad (r-10)(r+5) = 0$

$x(t) = c_1 e^{-5t} + c_2 e^{10t}$ $\quad y(t) = \frac{x' - x}{3}$

$y(t) = \frac{-6}{3} c_1 e^{-5t} + \frac{9}{3} c_2 e^{10t}$

ch7(c) $\begin{cases} x' = y \\ y' = 2z \\ z' = 0 \end{cases}$

$z = c_1, \quad y = 2c_1 t + c_2, \quad x = c_1 t^2 + c_2 t + c_3$

ch7(d) $\vec{u} = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2 + c_3 e^{\lambda_3 t} v_3$

$\Rightarrow$ eigenpairs are $\left(1, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right), \left(0, \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}\right), \left(-1, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}\right)$

ch8(e) $e^{0t} = 1$, so $e^{\lambda_1 t}, e^{\lambda_2 t}, e^{\lambda_3 t}$ all equal 1

Name KEY

Differential Equations and Linear Algebra 2250-1

Final Exam 7:30am 28 April 2008

Ch10. (Laplace Transform Methods)

It is assumed that you have memorized the basic 4-item Laplace integral table and know the 6 basic rules for Laplace integrals. No other tables or theory are required to solve the problems below. If you don't know a table entry, then leave the expression unevaluated for partial credit.

☐ [25%] Ch10(a): Apply Laplace's method to the system. Find a 2×2 system for $\mathcal{L}(x)$, $\mathcal{L}(y)$ [15%]. Solve it **only** for $\mathcal{L}(x)$, showing all solution steps [10%]. Do not solve for $x(t)$ or $y(t)$!

$$\begin{aligned} x'' &= 3y, \\ y'' &= 2x - y, \\ x(0) &= 0, \quad x'(0) = 0, \\ y(0) &= 0, \quad y'(0) = 1. \end{aligned}$$

ans: $\begin{pmatrix} s^2 & -3 \\ -2 & s^2+1 \end{pmatrix}\begin{pmatrix} Lx \\ Ly \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$Lx = \frac{3}{s^4+s^2-6}$

☐ [25%] Ch10(b): Find $f(t)$ by partial fraction methods, given

$$\mathcal{L}(f(t)) = \frac{8s^3 + 30s^2 + 32s + 40}{(s+2)^2(s^2+4)}.$$

ans: $f(t) = 3e^{-2t} + 4te^{-2t} + 5\cos 2t$

☐ [15%] Ch10(c): Solve for $f(t)$, given

$$\mathcal{L}(f(t)) = \frac{d}{ds}\left(\mathcal{L}\left(t^2 e^{3t}\right)\Big|_{s \to (s+3)}\right).$$

ans: $f(t) = -t^3$

☐ [20%] Ch10(d): Solve for $f(t)$, given

$$\mathcal{L}(f(t)) = \left(\frac{s+1}{s+2}\right)^2 \frac{1}{(s+2)^2}$$

ans: $f(t) = \left(t - t^2 + \frac{t^3}{6}\right)e^{-2t}$

☐ [15%] Ch10(e): Solve by Laplace's method for the solution $x(t)$:

$$x''(t) + 3x'(t) = 9e^{-3t}, \quad x(0) = x'(0) = 0.$$

ans: $x = 1 - e^{-3t} - 3te^{-3t}$

If you solved (a) through (e), then you have 100%. Doing (f) or (g) below erases all credit gained for either 10(b) or 10(d), as marked.

☐ [25%] Ch10(f): [Replaces 10(b)] Find $\mathcal{L}(f(t))$, given $f(t) = \sinh(2t)\frac{\sin(t)}{t}$.

$F(s) = \frac{\tan^{-1}(s+2) - \tan^{-1}(s-2)}{2}$

☐ [20%] Ch10(g): [Replaces 10(d)] Fill in the blank spaces in the Laplace table:

$f(t)$	t^3	e^{-2t}	$e^{-t}\cos(2t)$	$t\cos t$	t^2e^{2t}
$\mathcal{L}(f(t))$	$\frac{6}{s^4}$	$\frac{1}{s+2}$	$\frac{s+1}{s^2+2s+5}$	$-\frac{d}{ds}\left(\frac{s}{s^2+1}\right)$	$\frac{2}{(s-2)^3}$

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Ch10 S 2008 28 Apr

Ch10 (a) $s^2 \mathcal{L}(x) = 3\mathcal{L}(y) \rightarrow (s^2)\mathcal{L}(x) + (-3)\mathcal{L}(y) = 0$

$s^2\mathcal{L}(y) - 1 = 2\mathcal{L}(x) - \mathcal{L}(y) \rightarrow (-2)\mathcal{L}(x) + (s^2+1)\mathcal{L}(y) = 1$

$$\mathcal{L}(x) = \frac{\begin{vmatrix} 0 & -3 \\ 1 & s^2+1 \end{vmatrix}}{\begin{vmatrix} s^2 & -3 \\ -2 & s^2+1 \end{vmatrix}} = \frac{3}{s^4+s^2-6}$$

Ch10 (b) $\mathcal{L}(f) = \frac{a}{s+2} + \frac{b}{(s+2)^2} + \frac{cs+d}{s^2+4} = \mathcal{L}(ae^{-2t} + bte^{-2t} + c\cos 2t + \frac{d}{2}\sin 2t$

$8s^3+30s^2+32s+40 = a(s+2)(s^2+4) + b(s^2+4) + (cs+d)(s+2)^2$

$s=-2: \quad -64+120-64+40 = 0 + 8b + 0 \quad \boxed{b=4}$

$s=2i: \quad -64i+(-120)+64i+40 = 0 + 0 + (2ci+d)(4)(1+i)^2$

$-80 = 4(2ci+d)(i^2+2i+1)$

$-80 = 8(-2c+di)$

$-10 = -2c+di \quad \boxed{c=5,\ d=0}$

$s=0: \quad 0+0+0+40 = 8a+4b+4d$

$40 = 8a+16+0 \quad \boxed{a=3}$

Ch10 (c) $\mathcal{L}(f) = \frac{d}{ds}\mathcal{L}(te^{3t}e^{-3t}) = \mathcal{L}((-t)t^2e^{0t}) = \mathcal{L}(-t^3)$

Ch10 (d) $\mathcal{L}(f) = \frac{(s+1)^2}{(s+2)^4} = \frac{(x-2+1)^2}{x^4} = \left.\frac{(s-1)^2}{s^4}\right|_{s\to s+2}$

$= \left.\left(\frac{1}{s^2} - \frac{2}{s^3} + \frac{1}{s^4}\right)\right|_{s\to s+2} = \mathcal{L}\left(\left(t - t^2 + \frac{t^3}{6}\right)e^{-2t}\right)$

Ch10 (e) $(s^2+3s)Lx = \frac{9}{s+3} \rightarrow Lx = \frac{9}{s(s+3)^2} = \frac{1}{s} + \frac{-1}{s+3} + \frac{-3}{(s+3)^2}$

$x(t) = 1 - e^{-3t} + (-3)te^{-3t}$

Ch10 (f) $\mathcal{L}(tf(t)) = \mathcal{L}(\frac{1}{2}e^{2t}\sin t) - \mathcal{L}(\frac{1}{2}e^{-2t}\sin t)$

$-\frac{dF}{ds} = \frac{1}{2}\frac{1}{(s-2)^2+1} - \frac{1}{2}\frac{1}{(s+2)^2+1}$ Quadrature DE

$-2F = \tan^{-1}(s-2) - \tan^{-1}(s+2) + c \qquad 0 = \frac{\pi}{2} - \frac{\pi}{2} + c$

Ch10 (g) Standard table methods using the shift theorem and $\frac{d}{ds}\mathcal{L}(f) = \mathcal{L}((-t)f(t))$.