Differential Equations and Linear Algebra

2250-2 10:10am 25 April 2008

Instructions. The time allowed is 120 minutes. The examination consists of six problems, one for each of chapters 3, 4, 5, 6, 7, 10, each problem with multiple parts. A chapter represents 20 minutes on the final exam.

Each problem on the final exam represents several textbook problems numbered (a), (b), (c), $\cdots$. Choose the problems to be graded by check-mark $\boxed{\text{X}}$; the credits should add to 100. Each chapter (Ch3, Ch4, Ch5, Ch6, Ch7, Ch10) adds at most 100 towards the maximum final exam score of 600. The final exam score is reported as a percentage 0 to 100, which is the sum of the scores earned on six chapters divided by 600 to make a fraction, then converted to a percentage.

- Calculators, books, notes and computers are not allowed.
- Details count. Less than full credit is earned for an answer only, when details were expected. Generally, answers count only 25% towards the problem credit.
- Completely blank pages count 40% or less, at the whim of the grader.
- Answer checks are not expected or required. First drafts are expected, not complete presentations.
- Please submit **exactly six** separately stapled packages of problems, one package per chapter.

Final Grade. The final exam counts as two midterm exams. For example, if exam scores earned were 90, 91, 92 and the final exam score is 89, then the exam average for the course is

$$\text{Exam Average} = \frac{90 + 91 + 92 + 89 + 89}{5} = 90.2.$$

Dailies count 30% of the final grade. The course average is computed from the formula

$$\text{Course Average} = \frac{70}{100}(\text{Exam Average}) + \frac{30}{100}(\text{Dailies Average}).$$

Please discard this page or keep it for your records.

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Ch3. (Linear Systems and Matrices)

☐ [40%] Ch3(a): Let B be the invertible matrix given below, where $\boxed{?}$ means the value of the entry does not affect the answer to this problem. The second matrix C is the adjugate (or adjoint) of B. Let A be a matrix such that $C^T A^2 C^2 + B^2 C A^T = 0$. Find all possible values of $\det(A)$.

Notation: X^T is the transpose of X. And X^2 means XX.

$$B = \begin{pmatrix} 1 & -1 & -1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 1 & 0 & 0 & -3 \end{pmatrix}, \quad C = \begin{pmatrix} ? & ? & ? & 0 \\ 4 & -4 & 2 & ? \\ ? & -2 & -4 & 0 \\ ? & ? & ? & 5 \end{pmatrix}$$

ans: $\det A = 0$ or $\det(A) = \frac{1}{(15)^4}$

☐ [40%] Ch3(b): State the three possibilities for a linear system $A\mathbf{x} = \mathbf{b}$ [5%]. Determine which values of k correspond to these three possibilities, for the system $A\mathbf{x} = \mathbf{b}$ given in the display below [20%].

$$A = \begin{pmatrix} 2 & 3 & -k \\ 0 & k-2 & k-3 \\ 2 & 3 & -3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 2k-6 \\ k-3 \end{pmatrix}$$

ans:
NO sol $k-3=0$
∞-many $k-2=0$
unique $(k-2)(k-3) \neq 0$

☐ [20%] Ch3(c): Assume A is an $n \times n$ matrix and that $A\mathbf{x} = \mathbf{b}$ has no solution for any nonzero vector $\mathbf{b}$. Find a basis for the solution space of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

ans: a basis is the set of columns of the $n \times n$ identity.

If you solved (a), (b) and (c), then go on to Ch4. **Otherwise**, try (d), (e) and (f). Maximum credit is 100%. Problems below erase credit for problem Ch3(c) above - please read carefully!

☐ [10%] Ch3(d): [Replaces 1/2 of Ch3(c)] Give an example of a 3×3 matrix A such that the system $A\mathbf{x} = \mathbf{0}$ has a solution with $\|\mathbf{x}\| = 1$. The zero matrix works, $\vec{x} = \mathrm{col}(I, 1)$

☐ [10%] Ch3(e): [Replaces 1/2 of Ch3(c)] Prove or display a counterexample: a triangular 2×2 matrix is invertible. Counter example: The zero matrix is not invertible.

☐ [20%] Ch3(f): [Replaces all of Ch3(c)] Find the value of x_3 by Cramer's Rule in the system $C\mathbf{x} = \mathbf{b}$, given C and $\mathbf{b}$ below. Evaluate determinants by any hybrid method (triangular, swap, combo, multiply, cofactor). The use of 2×2 Sarrus' rule is allowed. The 3×3 Sarrus' rule is **disallowed**.

$$C = \begin{pmatrix} 0 & 2 & -1 & 0 \\ 0 & 0 & 4 & 0 \\ 1 & 3 & -2 & 1 \\ 0 & 0 & 2 & 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}$$

ans:
$x_3 = \frac{\Delta_3}{\Delta}$
$x_3 = \frac{2}{8} = \frac{1}{4}$

Staple this page to the top of all Ch3 work. Submit one package per chapter.

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Ch3(a) $BC = \det(B)I$ implies $\det(B) = \text{row}(B,4)\,\text{col}(C,4) = -15$

product Theorem $\Rightarrow$ $\det B \det C = (\det B)^4 \Rightarrow \det(C) = (-15)^3$

product Theorem $\Rightarrow$ $\det(C^T)\det(A)\det(A)\det(C)\det(C) = \det(-B)\det(B)\det(C)\det(A^T)$

$\det X^T = \det X$ $\Rightarrow$ $(-15)^3(\det A)^2(-15)^3(-15)^3 = \det(-I)(\det(B))^2\det(C)\det(A)$

$\Rightarrow (-15)^9(\det A)^2 = (-1)^4(-15)^2(-15)^3 \det A$

$\Rightarrow \det(A) = 0$ or $\det(A) = \dfrac{(-1)^4(-15)^5}{(-15)^9}$

Ch3(b) $\left(\begin{array}{ccc|c} 2 & 3 & -k & 1 \\ 0 & k-2 & k-3 & 2k-6 \\ 2 & 3 & -3 & k-3 \end{array}\right) \cong \left(\begin{array}{ccc|c} 2 & 3 & -k & 1 \\ 0 & k-2 & k-3 & 2k-6 \\ 0 & 0 & k-3 & k-4 \end{array}\right)$ combo(1,3,-1)

$\cong \left(\begin{array}{ccc|c} 2 & 3 & -k & 1 \\ 0 & k-2 & 0 & k-2 \\ 0 & 0 & k-3 & k-4 \end{array}\right)$ combo(3,2,-1)

NO Solution: $k-3=0$, because of signal eq "0=1"

∞-many solutions: $k-2=0$, because of one free variable

unique solution: $(k-2)(k-3) \neq 0$, because of 3 lead variables

Ch3(c) Let $\vec{x} = \text{col}(I,j)$ for some index j. Let $\vec{b} = A\vec{x}$. If $\vec{b} \neq \vec{0}$, then $A\vec{x} = \vec{b}$ has a solution $\vec{x}$, which violates the hypothesis. Hence $\vec{b} = \vec{0}$ and $\vec{x}$ is a sol of $A\vec{x} = \vec{0}$. All the cols of I are solutions and form a basis.

ch3(f) $x_3 = \dfrac{\Delta_3}{\Delta}$, $\Delta = \begin{vmatrix} 0 & 2 & -1 & 0 \\ 0 & 0 & 4 & 0 \\ 1 & 3 & -2 & 1 \\ 0 & 0 & 2 & 1 \end{vmatrix} \underset{\text{cofactor rule}}{=} (-1)(4)\begin{vmatrix} 0 & 2 & 0 \\ 1 & 3 & 1 \\ 0 & 0 & 1 \end{vmatrix} = (-1)^2(4)(2)\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}$

$\Delta = 8$

$\Delta_3 = \begin{vmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 3 & 2 & 1 \\ 0 & 0 & 3 & 1 \end{vmatrix} \underset{\text{cofactor rule}}{=} (-1)(2)\begin{vmatrix} 0 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & 1 \end{vmatrix} = (-1)^2(2)\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}$

$\Delta_3 = 2$

$x_3 = \dfrac{2}{8}$

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Ch4. (Vector Spaces)

☐ [25%] Ch4(a): Independence of 3 fixed vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ can be decided by counting the pivot columns of their augmented matrix. State a different test which can decide upon independence of three vectors [10%]. Apply one of these two tests and report for which values of x the three vectors are **independent** [15%].

$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 2 \\ 5 \\ 3 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} -2 \\ 2 \\ 9 \\ x \end{pmatrix}.$$

ans
[10%] Rank Test
Det Test
indep $\Leftrightarrow$ $x-6 \neq 0$

☐ [25%] Ch4(b): Consider the four vectors

$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}, \quad \mathbf{w}_1 = \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix}, \quad \mathbf{w}_2 = \begin{pmatrix} -1 \\ 3 \\ x \end{pmatrix}.$$

ans: $x+2=0$

Find one value of x such that the subspaces $S_1 = \mathbf{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ and $S_2 = \mathbf{span}\{\mathbf{w}_1, \mathbf{w}_2\}$ are equal, showing all details.

☐ [50%] Ch4(c): Define the 5×5 matrix A by the display below. Find a basis of fixed vectors in $\mathcal{R}^5$ for (1) the column space of A [25%] and (2) the row space of A [25%]. The two displayed bases **must** consist of columns of A and columns of A^T (the transpose of A), respectively.

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & -3 & -2 & 1 \\ 0 & -1 & -2 & -1 & 1 \\ 0 & 6 & 6 & 4 & 0 \\ 0 & 2 & 4 & 2 & 0 \end{pmatrix}$$

ans:
Colspace(A) = Span Cols 2,3,5
rowspace(A) = Span rows 2,3,4

If you finished (a), (b) and (c), then 100% has been marked – go on to Ch5. **Otherwise**, unmark one or more of (a), (b) or (c), then complete (d) and/or (e). A maximum of four problems will be graded.

☐ [25%] Ch4(d): Define S to be the set of all vectors $\mathbf{x}$ in $\mathcal{R}^4$ which are orthogonal to all the vectors $\mathbf{y}$ satisfying $B\mathbf{y} = \mathbf{0}$ for some 2×4 matrix B. Prove or disprove that S is a subspace of $\mathcal{R}^4$.

☐ [25%] Ch4(e): Find a 4×4 system of linear equations for the constants a, b, c, d in the partial fractions decomposition below [10%]. Solve for a, b, c, d, showing all solution steps [10%]. Report the answers [5%].

$$\frac{6x^3 + 6x^2 - 10x + 10}{(x-1)^2(x+1)^2} = \frac{a}{x-1} + \frac{b}{(x-1)^2} + \frac{c}{x+1} + \frac{d}{(x+1)^2}$$

ans: $a=2, b=3, c=4, d=5$

Staple this page to the top of all Ch4 work. Submit one package per chapter.

ch4 ⓐ Test 1: Rank Test, $\operatorname{rank}(\operatorname{aug}(v_1, v_2, v_3)) = 3 \Leftrightarrow$ indep.

Test 2: Determinant Test, $\det(\operatorname{aug}(v_1, v_2, v_3)) \neq 0$ $\Leftrightarrow$ indep, provided fixed vectors are in $\mathbb{R}^3$

use the rank Test:

$$A = \begin{pmatrix} 2 & 0 & -2 \\ 2 & 2 & 2 \\ 1 & 5 & 9 \\ 0 & 3 & x \end{pmatrix} \cong \begin{pmatrix} 2 & 0 & -2 \\ 0 & 2 & 4 \\ 1 & 5 & 9 \\ 0 & 3 & x \end{pmatrix} \cong \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 5 & 10 \\ 0 & 3 & x \end{pmatrix}$$

$$\cong \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 3 & x \end{pmatrix} \cong \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & x-6 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{Rank} = 3 \Leftrightarrow \boxed{x - 6 \neq 0}$$

ch4 ⓑ

- Already v_1, v_2 are indep, because one is not a multiple of the other.
- $$\operatorname{aug}(w_1, w_2) = \begin{pmatrix} 3 & -1 \\ -2 & 3 \\ -1 & x \end{pmatrix} \cong \begin{pmatrix} 3 & -1 \\ 1 & 2 \\ -1 & x \end{pmatrix} \cong \begin{pmatrix} 1 & 2 \\ 3 & -1 \\ -1 & x \end{pmatrix} \cong \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 0 & x+2 \end{pmatrix}$$

  $$\cong \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow w_1, w_2 \text{ independent.}$$
- Let $$C = \operatorname{aug}(v_1, v_2, w_1, w_2) = \begin{pmatrix} 2 & 1 & 3 & -1 \\ 1 & 2 & -2 & 3 \\ -3 & -3 & -1 & x \end{pmatrix}$$

$$\cong \begin{pmatrix} 1 & 2 & -2 & 3 \\ 2 & 1 & 3 & -1 \\ -3 & -3 & -1 & x \end{pmatrix} \text{swap} \cong \begin{pmatrix} 1 & 2 & -2 & 3 \\ 0 & -3 & 7 & -7 \\ -3 & -3 & -1 & x \end{pmatrix}$$

$$\cong \begin{pmatrix} 1 & 2 & -2 & 3 \\ 0 & -3 & 7 & -7 \\ 0 & 3 & -7 & x+9 \end{pmatrix} \cong \begin{pmatrix} 1 & 2 & -2 & 3 \\ 0 & -3 & 7 & -7 \\ 0 & 0 & 0 & x+2 \end{pmatrix}$$

$\operatorname{Rank}(C) = 2 \Leftrightarrow x + 2 = 0$

<u>Theorem</u> $\operatorname{span}\{v_1, v_2\} = \operatorname{span}\{w_1, w_2\} \Leftrightarrow \operatorname{rank}(A) = \operatorname{rank}(B) = \operatorname{rank}(C) =$ where $A = \operatorname{aug}(v_1, v_2)$, $B = \operatorname{aug}(w_1, w_2)$, $C = \operatorname{aug}(A, B)$

ch4 ⓒ A triangular form of A is $\begin{pmatrix} 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & -3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

pivots of $A = 2, 3, 5$

The $\operatorname{rref}(A^T) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

pivots of $A^T = 2, 3, 4$

ch 4 ⓓ $S = \{\vec{x} : \vec{x}\cdot\vec{y} = 0 \text{ for all } B\vec{y} = \vec{0}\}$

Let $\vec{y}_1, \dots, \vec{y}_k$ be a basis for the solution space of $B\vec{y} = \vec{0}$. Then

$$S = \{\vec{x} : \vec{x}\cdot\vec{y}_1 = 0, \dots, \vec{x}\cdot\vec{y}_k = 0\}$$

Let $A = \text{aug}(\vec{y}_1, \dots, \vec{y}_k)^T$. Then

$$S = \{\vec{x} : A\vec{x} = \vec{0}\}$$

S is a subspace by the kernel theorem

ch 4 ⓔ

$$6x^3 + 6x^2 - 10x + 10 = a(x-1)(x+1)^2 + b(x+1)^2 + c(x-1)^2(x+1) + d(x-1)^2$$

sample method to be used

$x = 1$: $12 = 0 + 4b + 0 + 0 \Rightarrow b = 3$

$x = -1$: $20 = 0 + 0 + 0 + 4d \Rightarrow d = 5$

$x = 0$: $10 = -a + b + c + d$

$10 = -a + 3 + c + 5$

$\underline{\underline{2 = -a + c}}$

$x = 2$: $6(8) + 6(4) - 20 + 10 = 9a + 9b + 3c + d$

$48 + 24 - 10 = 9a + 27 + 3c + 5$

$62 - 27 - 5 = 9a + 3c$

$30 = 9a + 3c$

$\underline{\underline{10 = 3a + c}}$

Solve to get $a = 2$, $c = 4$. ans: $\boxed{a = 2, b = 3, c = 4, d = 5}$

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Ch5. (Linear Equations of Higher Order)

☐ [10%] Ch5(a): Find the general solution $y(x)$ of the differential equation

$$3y'' + 22y' + 7y = 0.$$

ans : $y = c_1 e^{-x/3} + c_2 e^{-7x}$

☐ [20%] Ch5(b): Find the general solution of the homogeneous differential equation, given it has characteristic equation

$$(r^2 + 2r + 5)^2 (r^2 + 16)^3 = 0.$$

ans: y = l.c. of 10 atoms for roots $-1 \pm 2i$, $\pm 4i$

☐ [10%] Ch5(c): Given a damped spring-mass system $mx''(t) + cx'(t) + kx(t) = 0$ with $m = 3$, $c = 22$ and $k = 7$, classify the answer $x(t)$ as over-damped, critically damped or under-damped. Please, **do not solve** the differential equation!

☐ [30%] Ch5(d): Assume a ninth order constant-coefficient linear differential equation has characteristic equation $r^3(r^2+1)^2(r^2-r) = 0$. Suppose the right side of the differential equation is

$$f(x) = x^2(x + 2e^x) + 5x \sin x + \cos x$$

ans: 3 groups $s_1 = 4$, $s_2 = 1$, $s_3 = 2$ see solution

Determine the **corrected** trial solution for y_p according to the method of undetermined coefficients. **Do not evaluate** the undetermined coefficients!

☐ [10%] Ch5(e): Find a fifth order linear homogeneous differential equation with constant coefficients whose general solution is $y = c_1 + c_2 x + c_3 x^2 + c_4 \cos x + c_5 \sin x$. ans : $y^{v} + y''' = 0$

☐ [20%] Ch5(f): Find the steady-state periodic solution for the differential equation

$$x'' + 2x' + 17x = \cos(3t).$$

ans: $x(t) = \frac{2}{25} \cos 3t + \frac{3}{50} \sin 3t$

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ch5 (a) $3r^2+22r+7=0$
$(3r+1)(r+7)=0$ $\quad y = c_1 e^{-x/3} + c_2 e^{-7x}$

ch5 (b) $(r^2+2r+5)^2(r^2+16)^3 = ((r+1)^2+4)^2(r^2+16)^3$
roots $-1\pm 2i, -1\pm 2i, \pm 4i, \pm 4i, \pm 4i$
atoms = $e^{-x}\cos 2x,\ xe^{-x}\cos 2x,\ e^{-x}\sin 2x,\ xe^{-x}\sin 2x,$
$\cos 4x,\ x\cos 4x,\ x^2\cos 4x,\ \sin 4x,\ x\sin 4x,\ x^2\sin 4x$
y = l.c. of atoms, constants $c_1 \to c_{10}$

ch5 (c) over-damped by ch5 (a)

ch5 (d) $r^3(r^2+1)^2 r(r-1) = r^4(r-1)(r^2+1)^2$
roots = $0,0,0,0,1,\pm i,\pm i$

$$y = x^{s_1}(d_1 + d_2 x + d_3 x^2 + d_4 x^3) \qquad s_1 = 4$$
$$+ x^{s_2}(d_5 + d_6 x + d_7 x^2)e^x \qquad s_2 = 1$$
$$+ x^{s_3}(d_8\cos x + d_9 x\cos x + d_{10}\sin x + d_{11} x\sin x)$$
$$s_3 = 2$$

ch5 (e) char eq is $r^3(r^2+1) = r^5 + r^3$
DE is $y^{v} + y''' = 0$

ch5 (f) $x = d_1\cos 3t + d_2\sin 3t$
$x' = -3d_1\sin 3t + 3d_2\cos 3t$
$x'' = -9x$
$x'' + 2x' + 17x = \cos 3t$
$2x' + 8x = \cos 3t$
$(6d_2 + 8d_1)\cos 3t + (-6d_1 + 8d_2)\sin 3t = \cos 3t$

$$\begin{cases} 8d_1 + 6d_2 = 1 \\ -6d_1 + 8d_2 = 0 \end{cases} \qquad \Delta = \begin{vmatrix} 8 & 6 \\ -6 & 8 \end{vmatrix} = 64+36 = 100$$
$$\Delta_1 = \begin{vmatrix} 1 & 6 \\ 0 & 8 \end{vmatrix} = 8$$
$$\Delta_2 = \begin{vmatrix} 8 & 1 \\ -6 & 0 \end{vmatrix} = 6$$

$$d_1 = \frac{8}{100} = \frac{2}{25} \qquad d_2 = \frac{6}{100} = \frac{3}{50}$$

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Ch6. (Eigenvalues and Eigenvectors)

☐ [30%] Ch6(a): Find the eigenvalues of the matrix $A = \begin{pmatrix} 0 & 4 & -1 & 0 & 0 \\ -4 & 0 & -2 & 1 & 0 \\ 0 & 0 & 5 & 1 & 0 \\ 0 & 0 & 2 & 3 & 0 \\ 0 & 0 & 7 & 1 & 2 \end{pmatrix}$.

ans: $3, 4i, -4i, 4+\sqrt{3}, 4-\sqrt{3}$

To save time, **do not** find eigenvectors!

☐ [20%] Ch6(b): Consider a 3×3 real matrix A with eigenpairs

$$\left(5, \begin{pmatrix} 13 \\ 6 \\ -41 \end{pmatrix}\right), \quad \left(4i, \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix}\right), \quad \left(-4i, \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}\right).$$

(1) [10%] Display an invertible matrix P and a diagonal matrix D such that $AP = PD$.

(2) [10%] Display a matrix product formula for A, but do not evaluate the matrix products, in order to save time. ans: $A = PDP^{-1}$

☐ [15%] Ch6(c): Find a 2×2 matrix A with eigenpairs

$$\left(1, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right), \quad \left(-2, \begin{pmatrix} 1 \\ 4 \end{pmatrix}\right).$$

ans: $A = \begin{pmatrix} 4 & 3/2 \\ 12 & -5 \end{pmatrix}$

☐ [35%] Ch6(d): The matrix A below has eigenvalues 2, 5 and 0. Test A to see it is diagonalizable, and if it is, then display Fourier's model for A.

$$A = \begin{pmatrix} 4 & -2 & 1 \\ -1 & 3 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

☐ [20%] Ch6(e): If you were unable to earn 100% from problems (a) through (d), then solve this one, **otherwise proceed to Ch10**.
Assume A is a given 4×4 matrix with eigenvalues $0, 1, 2 \pm 3i$. Find the eigenvalues of $4A - 3I$, where I is the identity matrix. ans: $\lambda = 0, 1, 5 \pm 12i$

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Ch6(a) Expand $A-\lambda I$ by cofactors. Use col 1 to start

$$\det(A-\lambda I) = (-\lambda)\,\text{minor}(A-\lambda I,1,1) + (-1)(-4)\,\text{minor}(A-\lambda I,2,1)$$

$$= (-\lambda)\begin{vmatrix} -\lambda & -2 & 1 & 0 \\ 0 & 5-\lambda & 1 & 0 \\ 0 & 2 & 3-\lambda & 0 \\ 0 & 7 & 1 & 2-\lambda \end{vmatrix} + 4\begin{vmatrix} 4 & -1 & 0 & 0 \\ 0 & 5-\lambda & 1 & 0 \\ 0 & 2 & 3-\lambda & 0 \\ 0 & 7 & 1 & 2-\lambda \end{vmatrix}$$

$$= (-\lambda)(-\lambda)\begin{vmatrix} 5-\lambda & 1 & 0 \\ 2 & 3-\lambda & 0 \\ 7 & 1 & 3-\lambda \end{vmatrix} + 16\begin{vmatrix} 5-\lambda & 1 & 0 \\ 2 & 3-\lambda & 0 \\ 7 & 1 & 2-\lambda \end{vmatrix}$$

Both the same

$$= (\lambda^2+16)\begin{vmatrix} 5-\lambda & 1 & 0 \\ 2 & 3-\lambda & 0 \\ 7 & 1 & 3-\lambda \end{vmatrix} = (\lambda^2+16)(3-\lambda)(\lambda^2-8\lambda+13)$$

Roots of $\lambda^2-8\lambda+13 = \frac{8}{2} \pm \frac{1}{2}\sqrt{64-52} = 4 \pm \sqrt{3}$

Roots of $\lambda^2+16 = \pm 4i$

ans: $\boxed{3,\ 4i,\ -4i,\ 4+\sqrt{3},\ 4-\sqrt{3}}$

Ch6(b) $P = \begin{pmatrix} 13 & -i & i \\ 6 & 1 & 1 \\ -41 & 0 & 0 \end{pmatrix}$ $D = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 4i & 0 \\ 0 & 0 & -4i \end{pmatrix}$

$AP = PD \implies A = PDP^{-1}$

ch6(c) $A = PDP^{-1}$ where $P = \begin{pmatrix} 1 & 1 \\ 2 & 4 \end{pmatrix}$ $D = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$ $P^{-1} = \frac{1}{2}\begin{pmatrix} 4 & -1 \\ -2 & 1 \end{pmatrix}$

$A = \begin{pmatrix} 4 & 3/2 \\ 12 & -5 \end{pmatrix}$

ch6(d) Distinct eigenvalues $\implies$ diagonalizable

Eigenpairs $= \left(2, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right), \left(5, \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}\right), \left(0, \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}\right)$

$$A\left(c_1\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + c_3\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}\right) = 2c_1\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 5c_2\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + \underbrace{(0)c_3\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}}_{\text{zero vector}}$$

ch6(e) $4A - 3I - \lambda I = 4\left(A - \left(\frac{3}{4} + \frac{\lambda}{4}\right)I\right)$

$\implies \frac{3}{4} + \frac{\lambda}{4} = 0, 1, 2 \pm 3i$

$\implies \frac{\lambda}{4} = 0, \frac{1}{4}, \frac{5}{4} \pm 3i$

$\implies \lambda = 0, 1, 5 \pm 12i$

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Ch7. (Linear Systems of Differential Equations)

☐ [50%] Ch7(a): Apply the eigenanalysis method to solve the differential system $\mathbf{u}' = A\mathbf{u}$, given

ans:

$$\vec{u} = c_1 e^{2t}\begin{pmatrix}1\\1\\0\end{pmatrix} + c_2 e^{2t}\begin{pmatrix}-2\\0\\1\end{pmatrix} + c_3 e^{7t}\begin{pmatrix}-1\\0\\1\end{pmatrix} \quad A = \begin{pmatrix} -3 & 5 & -10 \\ 0 & 2 & 0 \\ 5 & -5 & 12 \end{pmatrix}$$

☐ [25%] Ch7(b): Solve for the general solution $x(t)$, $y(t)$ in the system below. Use any method that applies, from the four possible methods.

$$\frac{dx}{dt} = x + 2y,$$
$$\frac{dy}{dt} = -2x + 4y.$$

☐ [15%] Ch7(c): Solve the 3×3 differential system $\mathbf{u}' = A\mathbf{u}$ for matrix

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

☐ [10%] Ch7(d): Assume A is 2×2 and the general solution of $\mathbf{u}' = A\mathbf{u}$ is given by

$$\mathbf{u}(t) = c_1 e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

Display Fourier's model for A.

If you solved (a), (b), (c), (d), then you have marked 100%. If so, then go on to Ch10, otherwise, continue here. Selecting Ch7(e) erases all credit gained for Ch7(d). Only 4 parts will be graded.

☐ [10%] Ch7(e): [Replaces Ch7(d)] A 3×3 real matrix A has all eigenvalues equal to -1 and corresponding eigenvectors

$$\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ -5 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}.$$

Display the general solution of the differential equation $\mathbf{x}' = A\mathbf{x}$.

Staple this page to the top of all Ch7 work. Submit one package per chapter.

Ch7 S2008 25apr

ch7 (a) $(2-\lambda)(\lambda^2-9\lambda+14) = (2-\lambda)(\lambda-7)(\lambda-2)$

roots $2, 2, 7$

Eigenpairs $= \left(2, \begin{pmatrix}1\\1\\0\end{pmatrix}\right), \left(2, \begin{pmatrix}-2\\0\\1\end{pmatrix}\right), \left(7, \begin{pmatrix}-1\\0\\1\end{pmatrix}\right)$

$$\vec{u}(t) = c_1 e^{2t}\begin{pmatrix}1\\1\\0\end{pmatrix} + c_2 e^{2t}\begin{pmatrix}-2\\0\\1\end{pmatrix} + c_3 e^{7t}\begin{pmatrix}-1\\0\\1\end{pmatrix}$$

ch7 (b) $A = \begin{pmatrix}1 & 2\\ -2 & 4\end{pmatrix}$ $\lambda^2 - 5\lambda + 8 = 0$

roots $= \frac{5}{2} \pm \frac{\sqrt{7}}{2} i$ atoms $= e^{5t/2}\cos\frac{\sqrt{7}t}{2}, e^{5t/2}\sin\frac{\sqrt{7}t}{2}$

$x(t) = \left[c_1 \cos(\sqrt{7}t/2) + c_2 \sin(\sqrt{7}t/2)\right] e^{5t/2}$

$y(t) = \frac{x' - x}{2}$

$= \frac{e^{5t/2}}{4}\left((3c_1 + \sqrt{7}c_2)\cos\left(\frac{\sqrt{7}t}{2}\right) + (-\sqrt{7}c_1 + 3c_2)\sin\left(\frac{\sqrt{7}t}{2}\right)\right)$

ch7 (c) $\begin{cases} u_1' = 0 \\ u_2' = u_1 \\ u_3' = u_2 \end{cases}$

$$\boxed{\begin{aligned} u_1 &= c_1 \\ u_2 &= c_1 t + c_2 \\ u_3 &= c_1 \frac{t^2}{2} + c_2 t + c_3 \end{aligned}}$$

ch7 (d) $A\left(c_1\begin{pmatrix}1\\1\end{pmatrix} + c_2\begin{pmatrix}-2\\1\end{pmatrix}\right) = c_1\begin{pmatrix}1\\1\end{pmatrix} + 0\cdot c_2\begin{pmatrix}-2\\1\end{pmatrix}$

ch7 (e) $\vec{x}(t) = c_1 e^{-t}\begin{pmatrix}0\\-1\\1\end{pmatrix} + c_2 e^{-t}\begin{pmatrix}0\\-5\\1\end{pmatrix} + c_3 e^{-t}\begin{pmatrix}1\\-1\\2\end{pmatrix}$

Name KEY

Differential Equations and Linear Algebra 2250-2

Final Exam 10:10am 25 April 2008

Ch10. (Laplace Transform Methods)

It is assumed that you have memorized the basic 4-item Laplace integral table and know the 6 basic rules for Laplace integrals. No other tables or theory are required to solve the problems below. If you don't know a table entry, then leave the expression unevaluated for partial credit.

☐ [25%] Ch10(a): Apply Laplace's method to the system. Find a 2×2 system for $\mathcal{L}(x)$, $\mathcal{L}(y)$ [15%]. Solve it **only** for $\mathcal{L}(x)$, showing all solution steps [10%]. Do not solve for $x(t)$ or $y(t)$!

$$\begin{aligned} x'' &= 3y, \\ y'' &= 2x - y, \\ x(0) &= 0, \quad x'(0) = 0, \\ y(0) &= 1, \quad y'(0) = 0. \end{aligned}$$

ans: $\mathcal{L}(x) = \dfrac{3s}{s^4+s^2-6}$

☐ [25%] Ch10(b): Find $f(t)$ by partial fraction methods, given

$$\mathcal{L}(f(t)) = \frac{3s^3 - 5s^2 + 12s - 4}{(s-2)^2(s^2+4)}.$$

ans: $f = 2e^{2t} + 3te^{2t} + \cos(2t)$

☐ [15%] Ch10(c): Solve for $f(t)$, given

$$\mathcal{L}(f(t)) = \left(\left(\frac{d}{ds} \right) \left(\mathcal{L}(t^2 e^{3t}) \right) \right) \Bigg|_{s \to (s+3)}.$$

ans: $f(t) = -t^3$

☐ [20%] Ch10(d): Solve for $f(t)$, given

$$\mathcal{L}(f(t)) = \frac{(s-1)^2}{(s+2)^4}$$

ans: $(t - 3t^2 + \frac{3}{2}t^3)e^{-2t}$

☐ [15%] Ch10(e): Solve by Laplace's method for the solution $x(t)$:

$$x''(t) + 2x'(t) = e^{-2t}, \quad x(0) = x'(0) = 0.$$

ans: $x(t) = \frac{1}{4} - \frac{1}{2}e^{-2t} - \frac{1}{4}te^{-2t}$

If you solved (a) through (e), then you have 100%. Doing (f) or (g) below erases all credit gained for either 10(b) or 10(d), as marked.

☐ [25%] Ch10(f): [Replaces 10(b)] Find $\mathcal{L}(f(t))$, given $f(t) = \sinh(t)\dfrac{\sin(t)}{t}$.

ans: $\mathcal{L}(f) = \dfrac{\tan^{-1}(s+1) - \tan^{-1}(s-}{2}$

☐ [20%] Ch10(g): [Replaces 10(d)] Fill in the blank spaces in the Laplace table:

$f(t)$	t^3	e^{2t}	$\cos(\sqrt{7}t)$	$t \sin t$	$t^2 e^t$
$\mathcal{L}(f(t))$	$\dfrac{6}{s^4}$	$\dfrac{1}{s-2}$	$\dfrac{s}{s^2+7}$	$\dfrac{2s}{(s^2+1)^2}$	$\dfrac{2}{(s-1)^3}$

Staple this page to the top of all Ch10 work. Submit one package per chapter.

Ch10 ⓐ $s^2\mathcal{L}(x) = 3\mathcal{L}(y) \longrightarrow$ Eq1: $(s^2)\mathcal{L}(x)+(-3)\mathcal{L}(y)=0$

$s^2\mathcal{L}(y)-s = 2\mathcal{L}(x)-\mathcal{L}(y) \longrightarrow$ Eq2: $(-2)\mathcal{L}(x)+(s^2+1)\mathcal{L}(y)=s$

$A = \begin{pmatrix} s^2 & -3 \\ -2 & s^2+1 \end{pmatrix}$, $\Delta = s^4+s^2-6$, $\Delta_1 = \begin{vmatrix} 0 & -3 \\ s & s^2+1 \end{vmatrix} = 3s$

$\mathcal{L}(x) = \dfrac{\Delta_1}{\Delta} = \boxed{\dfrac{3s}{s^4+s^2-6}}$

Ch10 ⓑ $\mathcal{L}(f) = \dfrac{a}{s-2} + \dfrac{b}{(s-2)^2} + \dfrac{cs+d}{s^2+4}$

$= \mathcal{L}\left(ae^{2t} + bte^{2t} + c\cos(2t) + \frac{d}{2}\sin(2t)\right)$

clear fractions:

$3s^3-5s^2+12s-4 = a(s-2)(s^2+4) + b(s^2+4) + (cs+d)(s-2)^2$

<u>Sampling method</u>

$s=2$: $24-20+24-4 = 0 + 8b + 0 \longrightarrow \boxed{b=3}$

$s=2i$: $24i^3-5(-4)+24i-4 = 0+0+(2ci+d)(2i-2)^2$

$16 = 4(2ci+d)(i-1)^2$

$4 = (2ci+d)(i^2-2i+1)$

$4 = (2ci+d)(-2i)$

$-2 = (2ci+d)(i)$

$-2 = -2c + di$

$\begin{cases} -2c = -2 \\ d = 0 \end{cases} \longrightarrow \boxed{c=1,\ d=0}$

$s=0$: $0+0+0-4 = -8a+4b+4d$

$-4 = -8a+12+0 \longrightarrow \boxed{a=2}$

$-16 = -8a$

$\boxed{f(t) = 2e^{2t} + 3te^{2t} + \cos(2t)}$

Ch10 (c) $\mathcal{L}(f) = \left(\frac{d}{ds} \mathcal{L}(t^2 e^{3t}) \right) \Big|_{s \to s+3}$

$= \left(\mathcal{L}((-t) t^2 e^{3t}) \right) \Big|_{s \to s+3}$

$= \mathcal{L}(e^{-3t} (-t) t^2 e^{3t})$

$= \mathcal{L}(-t^3)$

$\boxed{f(t) = -t^3}$

ch10 (d) $\mathcal{L}(f) = \frac{(s-1)^2}{(s+2)^4}$ $x = s+2$

$= \frac{(x-2-1)^2}{x^4}$

$= \frac{x^2 - 6x + 9}{x^4}$

$= \left(\frac{1}{s^2} - \frac{6}{s^3} + \frac{9}{s^4} \right) \Big|_{s \to s+2}$

$= \mathcal{L}(t - 3t^2 + \frac{3}{2} t^3) \Big|_{s \to s+2}$

$= \mathcal{L}(e^{-2t} (t - 3t^2 + \frac{3}{2} t^3))$

$\boxed{f(t) = (t - 3t^2 + \frac{3}{2} t^3) e^{-2t}}$

ch10 (e) $(s^2 + 2s) \mathcal{L}(x) = \mathcal{L}(e^{-2t})$

$\mathcal{L}(x) = \frac{1}{s(s+2)^2}$

$= \frac{a}{s} + \frac{b}{s+2} + \frac{c}{(s+2)^2}$

$= \mathcal{L}(a + b e^{-2t} + c t e^{-2t})$

$x(t) = a + b e^{-2t} + c t e^{-2t}$

clear fractions: $1 = a(s+2)^2 + bs(s+2) + cs$

$s=0$: $1 = 4a + 0 + 0 \longrightarrow$

$s=-2$: $1 = 0 + 0 - 2c \longrightarrow$

$s=-1$: $1 = a - b - c \longrightarrow$

$\boxed{\begin{array}{l} a = 1/4 \\ c = -1/2 \\ b = -1/4 \end{array}}$

Ch10 (f) $\quad t f(t) = \sinh(t) \sin(t)$

$$2t f(t) = e^{t} \sin t - e^{-t} \sin t$$

$$2\mathcal{L}(t f(t)) = \mathcal{L}(e^{t} \sin t) - \mathcal{L}(e^{-t} \sin t)$$

$$-2 \frac{d}{ds} F(s) = \frac{1}{(s-1)^2+1} - \frac{1}{(s+1)^2+1}$$

$$-2F(s) = \tan^{-1}(s-1) - \tan^{-1}(s+1) + C$$

Take limit $s = \infty$, use Thm $\lim\limits_{s\to\infty} F(s) = 0$ to get C:

$$0 = \frac{\pi}{2} - \frac{\pi}{2} + C$$

ans : $\boxed{\mathcal{L}(f(t)) = \dfrac{\tan^{-1}(s-1) - \tan^{-1}(s+1)}{-2}}$

Ch10 (g) $\quad \mathcal{L}(t \sin t) = -\frac{d}{ds} \mathcal{L}(\sin t)$

$$= -\frac{d}{ds} \frac{1}{s^2+1}$$

$$= \boxed{\frac{2s}{(s^2+1)^2}}$$

$$\mathcal{L}(t^2 e^{t}) = \mathcal{L}(t^2) \Big|_{s \to s-1}$$

$$= \frac{2}{s^3} \Big|_{s \to s-1}$$

$$= \boxed{\frac{2}{(s-1)^3}}$$