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# Applied Differential Equations 2250

Exam date: Tuesday, 15 April, 2008

**Instructions**: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%.

1. (ch4) Complete enough of the following to add to 100%.

(a) [100%] Let $V$ be the vector space of all continuous functions defined on $0 \le x \le 1$. Define $S$ to be the set of all infinitely differentiable functions $f(x)$ in $V$ such that $f'(0) = 2f(0)$ and $f''(x)+2f'(x)+5f(x) = 0$. Prove that $S$ is a subspace of $V$.

(b) [50%] **If you solved (a), then skip (b) and (c)**. Let $V$ be the set of all $4 \times 1$ column vectors $\vec{x}$ with components $x_1$, $x_2$, $x_3$, $x_4$. Assume the usual $\mathcal{R}^4$ rules for addition and scalar multiplication. Let $S$ be the subset of $V$ defined by the equations

$$x_1 + x_3 = 0, \quad x_2 = x_4, \quad \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 3 & 0 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Prove that $S$ is a subspace of $V$.

(c) [50%] **If you solved (a), then skip (b) and (c)**. Solve for the unknowns $x_1$, $x_2$, $x_3$, $x_4$ in the system of equations below by showing all details of a frame sequence from the augmented matrix $C$ to **rref**$(C)$. Report the **vector form** of the general solution.

$$\begin{array}{rcrcrcrcl} x_1 & - & 5x_2 & + & 4x_3 & + & x_4 & = & 8 \\ x_1 & - & 2x_2 & - & 2x_3 & + & x_4 & = & 5 \\ & - & x_2 & + & 2x_3 & & & = & 1 \\ x_1 & - & 3x_2 & & & + & x_4 & = & 6 \end{array}$$

(a) $r^2+2r+5=0$ has roots $-1\pm 2i$. Then $f(x) = c_1 e^{-x}\cos 2x + c_2 e^{-x}\sin 2x$ is $\infty$-differentiable, hence in $V$. Condition $f'(0) = 2f(0)$ means $-c_1+2c_2 = 2c_1$. Then $S = \{c\, g(x) : c \text{ arbitrary}\}$ where $g(x) = e^{-x}\cos 2x + (3/2)e^{-x}\sin 2x$. The subspace criterion: (1) $\vec{0}$ is in $S$ because $\vec{0}$ equals $0 \cdot g(x)$; (2) If $ag$ and $bg$ are in $S$ then $ag+bg = cg$ where $c = a+b$, is in $S$; (3) If $ag$ is in $S$ and $k$ = constant, then $k(ag) = (ka)g = cg$ is in $S$, $c = ka$. Then $S$ is a subspace of $V$.

(b) Apply the kernel theorem to $A = \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 3 & 0 & -3 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$

(c) $C = \left(\begin{array}{cccc|c} 1 & -5 & 4 & 1 & 8 \\ 1 & -2 & -2 & 1 & 5 \\ 0 & -1 & 2 & 0 & 1 \\ 1 & -3 & 0 & 1 & 6 \end{array}\right)$, $\text{rref}(C) = \begin{pmatrix} 1 & 0 & -6 & 1 & 3 \\ 0 & 1 & -2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$, $\vec{x} = \begin{pmatrix} 3 \\ -1 \\ 0 \\ 0 \end{pmatrix} + t_1 \begin{pmatrix} 6 \\ 2 \\ 1 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

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**2.** **(ch5)** Complete (a), (b) and then either (c) or (d). **Do not do both** (c) and (d).

(a) [30%] Given $2x''(t) + 6x'(t) + 2wx(t) = 0$, which represents a damped spring-mass system with $m = 2$, $c = 6$, $k = 2w$, determine all values of $w$ such that the equation is over-damped [10%], critically damped [10%] or under-damped [10%].

(b) [40%] Find a particular solution $y_p(x)$ and the homogeneous solution $y_h(x)$ for $y^{iv} + 9y'' = 324x^2$. Reminder: $y^{iv}$ is the fourth derivative.

(c) [30%] Find by undetermined coefficients the steady-state periodic solution for the forced spring-mass system $x'' + 2x' + 17x = 130\sin(t)$.

(d) [30%] **If you did (c) above, then skip this one!** Find by variation of parameters a particular solution $y_p(x)$ for the equation $2y'' + 3y' = 18x$.

(a) $r^2 + 3r + w = 0$. overdamped $\iff 9 - 4w > 0$; critically damped $\iff 9 - 4w = 0$; underdamped $\iff 9 - 4w < 0$.

(b) $r^2(r^2+9) = 0 \implies y_h = c_1 + c_2 x + c_3 \cos 3x + c_4 \sin 4x$

$y_p = (d_1 + d_2 x + d_3 x^2)x^2$ by the fixup rule. Then $d_1 = -4, d_2 = 0, d_3 = 3$

(c) $x_{ss}(t) = 8 \sin t - \cos t$. Find it as $x = d_1 \cos t + d_2 \sin t$. Then $x'' = -x$ and $2x' + 16x = 130 \sin t$. Substitute to get equations

$$\begin{cases} 16 d_1 + 2d_2 = 0 \\ -2 d_1 + 16 d_2 = 130 \end{cases}$$

Solve by Cramer's rule $d_1 = \dfrac{\Delta_1}{\Delta}$, $d_2 = \dfrac{\Delta_2}{\Delta}$, where $\Delta = \begin{vmatrix} 16 & 2 \\ -2 & 16 \end{vmatrix} = 260$,

$\Delta_1 = \begin{vmatrix} 0 & 2 \\ 130 & 16 \end{vmatrix} = -260$, $\Delta_2 = \begin{vmatrix} 16 & 0 \\ -2 & 130 \end{vmatrix} = (130)(16)$. Then $d_1 = -1$, $d_2 = 8$.

(d) $2r^2 + 3r = 0 \implies$ atoms $1, e^{-3x/2}$ are symbols $y_1, y_2$ in (33) of section 5.5 of E&P. Then $f(x) = 18x$ and

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = -\frac{3}{2} e^{-3x/2}$$

(33) $y_p = u_1 y_1 + u_2 y_2$

$u_1 = -\int \dfrac{y_2 f}{2W} = 6x^2/2$

$u_2 = \int \dfrac{y_1 f}{2W} = (-8x + \frac{16}{3}) e^{3x/2}/2$

$y_p = 3x^2 - 4x + \frac{8}{3}$

• Best answer is $y_p = 3x^2 - 4x$ because $\frac{8}{3}$ is a solution of the homog DE.

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**3.** **(ch5)** Complete all parts below.

(a) [40%] A non-homogeneous linear differential equation with constant coefficients has right side $f(x) = x^2e^{-x} + x^3(x+5) + e^x \sin 2x + \cos 2x$ and characteristic equation of order 9 with roots $0, 0, 0, 0,\ 1,\ -1, -1,\ 2i, -2i$, listed according to multiplicity. Determine the undetermined coefficients **corrected** trial solution for $y_p$ according to the **fixup rule**. To save time, **do not** evaluate the undetermined coefficients and **do not** find $y_p(x)$! Undocumented detail or guessing earns no credit.

(b) [20%] The general solution of a linear differential equation with constant coefficients is

$$y = c_1 \cos 3x + c_2 \sin 3x + (c_3 + c_4 x)e^{2x} + c_5.$$

Find the roots of the characteristic equation.

(c) [20%] Find ~~five~~ Four independent solutions of the homogeneous linear constant coefficient equation whose ~~fifth~~ fourth order characteristic equation has roots $\pi,\ 0,\ 2+3i,\ 2-3i$.

(d) [20%] Assume $f(x)$ is a nonzero solution of $y'' + 4y = 0$. Find the corrected trial solution in the method of undetermined coefficients for the differential equation $y'' + 4y = f(x)$. To save time, **do not** evaluate the undetermined coefficients and **do not** find $y_p(x)$!

Fixup rule applied ✓

(a)
$$\begin{aligned} y_p = {} & x^{s_1}(d_1 + d_2 x + d_3 x^2)e^{-x} \\ & + x^{s_2}(d_4 + d_5 x + d_6 x^2 + d_7 x^3 + d_8 x^4) \\ & + x^{s_3}(d_9 \cos 2x + d_{10} \sin 2x)e^{x} \\ & + x^{s_4}(d_{11} \cos 2x + d_{12} \sin 2x) \end{aligned}$$

$s_1 = 2$

$s_2 = 4$

$s_3 = 0$

$s_4 = 1$

(b) Roots $= \pm 3i,\ 2, 2,\ 0$

(c) $e^{\pi x},\ 1,\ e^{2x}\cos 3x,\ e^{2x}\sin 3x$

(d) $y_p = (d_1 \cos 2x + d_2 \sin 2x)\, x$

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**4.** **(ch6)** Complete all of the items below.

(a) [30%] Find the eigenvalues of the matrix $A = \begin{pmatrix} -2 & 7 & 1 & 19 \\ -1 & 6 & -3 & 21 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & -1 & 0 \end{pmatrix}$. To save time, **do not** find eigenvectors!

(b) [30%] Assume $A$ is $2 \times 2$ and Fourier's model holds:

$$A\left(c_1\begin{pmatrix}1\\-1\end{pmatrix} + c_2\begin{pmatrix}1\\1\end{pmatrix}\right) = 2c_1\begin{pmatrix}1\\-1\end{pmatrix}.$$

Find $A$.

(c) [40%] Let $A = \begin{pmatrix} -1 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. Circle the possible eigenvectors of $A$ in the list below.

$$\begin{pmatrix}0\\0\\2\end{pmatrix}, \boxed{\begin{pmatrix}2\\0\\0\end{pmatrix}}, \boxed{\begin{pmatrix}4\\2\\0\end{pmatrix}}.$$

(a) Expand $\det(A-\lambda I)$ by cofactors along column one. Then

$\det(A-\lambda I) = (-2-\lambda)(6-\lambda)\Delta + (-1)(-1)(7)\Delta$, $\Delta = \begin{vmatrix} 3-\lambda & 2 \\ -1 & -\lambda \end{vmatrix}$

Then $\Delta = 0$ or else $(-2-\lambda)(6-\lambda)+7=0$.

The roots $= -1, 1, 2, 5$

(b) $P = \begin{pmatrix}1 & 1\\-1 & 1\end{pmatrix}$, $D = \begin{pmatrix}2 & 0\\0 & 0\end{pmatrix}$, $AP = PD \Rightarrow A = PDP^{-1}$

$A = \begin{pmatrix}1 & 1\\-1 & 1\end{pmatrix}\begin{pmatrix}2 & 0\\0 & 0\end{pmatrix}\begin{pmatrix}1 & -1\\1 & 1\end{pmatrix}\frac{1}{2} = \boxed{\begin{pmatrix}1 & -1\\-1 & 1\end{pmatrix}}$

(c) Test $A\vec{x} = \lambda\vec{x}$ for each $\vec{x}$ in the list.

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5. **(ch6)** Complete all parts below.

Consider the $3 \times 3$ matrix

$$E = \begin{pmatrix} 4 & 2 & -2 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix}.$$

Matrix $E$ has a Fourier model:

$$E\left(c_1\begin{pmatrix}1\\0\\0\end{pmatrix} + c_2\begin{pmatrix}0\\1\\1\end{pmatrix} + c_3\begin{pmatrix}2\\-1\\1\end{pmatrix}\right) = 4c_1\begin{pmatrix}1\\0\\0\end{pmatrix} + 4c_2\begin{pmatrix}0\\1\\1\end{pmatrix} + 2c_3\begin{pmatrix}2\\-1\\1\end{pmatrix}.$$

(a) [40%] Find $E^3\begin{pmatrix}3\\0\\2\end{pmatrix}$ without using matrix multiply.

(b) [30%] Let $P = \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix}$, $D = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}$ and define $A$ by the equation $AP = PD$. Display the eigenpairs of $A$.

(c) [30%] Assume the vector general solution $\mathbf{x}(t)$ of the linear differential system $\mathbf{x}' = M\mathbf{x}$ is given by

$$\mathbf{x}(t) = c_1 e^{-t}\begin{pmatrix}2\\1\end{pmatrix} + c_2 e^{2t}\begin{pmatrix}-1\\1\end{pmatrix}.$$

Display an invertible matrix $P$ and a diagonal matrix $D$ such that $MP = PD$.

(a) $E^3\vec{x} = c_1\lambda_1^3 v_1 + c_2\lambda_2^3 v_2 + c_3\lambda_3^3 v_3$

$\vec{x} = \begin{pmatrix}3\\0\\2\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix} + \begin{pmatrix}0\\1\\1\end{pmatrix} + \begin{pmatrix}2\\-1\\1\end{pmatrix}$ implies $c_1 = c_2 = c_3 = 1$. Then

$E^3\vec{x} = 4^3\begin{pmatrix}1\\0\\0\end{pmatrix} + 4^3\begin{pmatrix}0\\1\\1\end{pmatrix} + 2^3\begin{pmatrix}2\\-1\\1\end{pmatrix} = \boxed{\begin{pmatrix}80\\56\\72\end{pmatrix}}$

(b) $\left(3, \begin{pmatrix}3\\1\end{pmatrix}\right)$ and $\left(-2, \begin{pmatrix}1\\-1\end{pmatrix}\right)$

(c) $P = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$

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