Name KEY Classtime ______ 2250 Midterm 3 Ver 1 [7:30, S2008]

Applied Differential Equations 2250

Exam date: Tuesday, 15 April, 2008

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%.

1. **(ch4)** Complete enough of the following to add to 100%.

(a) [100%] Let V be the vector space of all continuous functions defined on $0 \le x \le 1$. Define S to be the set of all twice-continuously differentiable functions $f(x)$ in V such that $f'(0) = f(0)$ and $f''(x) + 3f'(x) + 2f(x) = 0$. Prove that S is a subspace of V.

(b) [50%] **If you solved (a), then skip (b) and (c).** Let V be the set of all 4×1 column vectors $\vec{x}$ with components x_1, x_2, x_3, x_4. Assume the usual $\mathcal{R}^4$ rules for addition and scalar multiplication. Let S be the subset of V defined by the equations

$$x_1 + x_2 = 0, \quad x_3 = x_4, \quad \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 0 & 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Prove that S is a subspace of V.

(c) [50%] **If you solved (a), then skip (b) and (c).** Solve for the unknowns x_1, x_2, x_3, x_4 in the system of equations below by showing all details of a frame sequence from the augmented matrix C to **rref**(C). Report the **vector form** of the general solution.

$$\begin{array}{rcrcrcrcr} x_1 & + & 10x_2 & + & 4x_3 & + & x_4 & = & 8 \\ x_1 & + & 4x_2 & - & 2x_3 & + & x_4 & = & 5 \\ & & 2x_2 & + & 2x_3 & & & = & 1 \\ x_1 & + & 6x_2 & & & + & x_4 & = & 6 \end{array}$$

(a) $r^2+3r+2=0$, $(r+1)(r+2)=0$, Then $S = \{c_1e^{-x}+c_2e^{-2x} : f'(0)=f(0)\}$ $= \{c_1e^{-x}+c_2e^{-2x} : -c_1-2c_2 = c_1+c_2\} = \{c_1e^{-x}+c_2e^{-2x} : 2c_1+3c_2=0\}$ $= \{c_1e^{-x} - \frac{2c_1}{3}e^{-2x} : c_1 \text{ arbitrary}\}$. All elements of S are ∞-differentiable, and constant multiples of $g(x) = 3e^{-x}-2e^{-2x}$. Then $S = \{c\,g(x) : c \text{ arbitrary}\} = \text{span}\{g\}$. Subspace criterion, (1) $\vec{0}$ in S because $\vec{0} = 0\cdot g$; (2) if ag and bg are in S then $ag+bg = cg$ with $c=a+b$ is in S; (3) If cg is in S, then $d(cg) = (dc)g$ is in S. Then S is a subspace of V

(b) Apply the kernel theorem to $A = \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 0 & 2 & -2 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$. Then S is a subspace of V.

(c) $C = \left(\begin{array}{cccc|c} 1 & 10 & 4 & 1 & 8 \\ 1 & 4 & -2 & 1 & 5 \\ 0 & 2 & 2 & 0 & 1 \\ 1 & 6 & 0 & 1 & 6 \end{array}\right)$, $\text{rref}(C) = \begin{pmatrix} 1 & 0 & -6 & 1 & 3 \\ 0 & 1 & 1 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$; $\vec{x} = \begin{pmatrix} 3 \\ 1/2 \\ 0 \\ 0 \end{pmatrix} + t_1\begin{pmatrix} 6 \\ -2 \\ 1 \\ 0 \end{pmatrix} + t_2\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

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Name KEY Classtime ______ 2250 Midterm 3 Ver 1 [7:30, S2008]

2. **(ch5)** Complete (a), (b) and then either (c) or (d). **Do not do both** (c) and (d).

(a) [30%] Given $4x''(t) + 20x'(t) + 4wx(t) = 0$, which represents a damped spring-mass system with $m = 4$, $c = 20$, $k = 4w$, determine all values of w such that the equation is over-damped [10%], critically damped [10%] or under-damped [10%].

(b) [40%] Find a particular solution $y_p(x)$ and the homogeneous solution $y_h(x)$ for $y^{iv} + 16y'' = 96x$. Reminder: y^{iv} is the fourth derivative.

(c) [30%] Find by undetermined coefficients the steady-state periodic solution for the forced spring-mass system $x'' + 2x' + 10x = 170\sin(t)$.

(d) [30%] **If you did (c) above, then skip this one!** Determine the practical resonance frequency ω for the equation $x'' + 2x' + 10x = 100\cos(\omega t)$.

(a) $r^2 + 5r + w = 0$ $\quad 25 - 4w > 0 \Rightarrow$ overdamped, $25 - 4w = 0 \Rightarrow$ critically damped, $25 - 4w < 0 \Rightarrow$ underdamped.

(b) $r^2(r^2 + 16) = 0 \Rightarrow \quad y_h = c_1 + c_2 x + c_3 \cos 4x + c_4 \sin 4x$

$y_p = (d_1 + d_2 x)x^2$ with $d_1 = 0, d_2 = 1$ giving $y_p = x^3$

(c) use $x = d_1 \cos t + d_2 \sin t$, $x' = -d_1 \sin t + d_2 \cos t$, $x'' = -x$ to get

$$\begin{cases} 9d_1 + 2d_2 = 0 \\ -2d_1 + 9d_2 = 170 \end{cases}$$

Then $\Delta = \begin{vmatrix} 9 & 2 \\ -2 & 9 \end{vmatrix} = 85$, $\Delta_1 = \begin{vmatrix} 0 & 2 \\ 170 & 9 \end{vmatrix} = -4(85)$, $\Delta_2 = \begin{vmatrix} 9 & 0 \\ -2 & 170 \end{vmatrix} = 18(85)$

gives $d_1 = \frac{\Delta_1}{\Delta} = -4$ and $d_2 = \frac{\Delta_2}{\Delta} = 18$

$$x(t) = -4\cos t + 18 \sin t$$

(d) use $\omega = \sqrt{\frac{k}{m} - \frac{c^2}{2m^2}} = \sqrt{\frac{10}{1} - \frac{4}{2}} = \sqrt{8}$

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Name KEY Classtime ______ 2250 Midterm 3 Ver 1 [7:30, S2008]

3. **(ch5)** Complete all parts below.

(a) [40%] A non-homogeneous linear differential equation with constant coefficients has right side $f(x) = x^2e^{-x} + x^3(x+4) + e^x \cos 2x$ and characteristic equation of order 8 with roots $0, 0, 0,\ 1,\ -1, -1,\ 2i, -2i,$ listed according to multiplicity. Determine the undetermined coefficients **corrected** trial solution for y_p according to the **fixup rule**. To save time, **do not** evaluate the undetermined coefficients and **do not** find $y_p(x)$! Undocumented detail or guessing earns no credit.

(b) [20%] The general solution of a linear differential equation with constant coefficients is

$$y = c_1 \cos 2x + c_2 \sin 2x + (c_3 + c_4 x)e^x + c_5.$$

Find the roots of the characteristic equation.

(c) [20%] Find ~~seven~~ six independent solutions of the homogeneous linear constant coefficient equation whose ~~seventh~~ sixth order characteristic equation has roots $-1,\ 0,\ 0,\ 0,\ 2+3i,\ 2-3i$.

(d) [20%] Assume $f(x)$ is a nonzero solution of $y'' + y = 0$. Find the corrected trial solution in the method of undetermined coefficients for the differential equation $y'' + y = f(x)$. To save time, **do not** evaluate the undetermined coefficients and **do not** find $y_p(x)$!

(a) $y = x^{s_1}(d_1 e^{-x} + d_2 x e^{-x} + d_3 x^2 e^{-x})$
$+ x^{s_2}(d_4 + d_5 x + d_6 x^2 + d_7 x^3 + d_8 x^4)$
$+ x^{s_3}(d_9 e^x \cos 2x + d_{10} e^x \sin 2x)$

Fixup rule says $s_1 = 2,\ s_2 = 3,\ s_3 = 0$

(b) Roots $= 2i, -2i, 1, 1, 0$

(c) $e^{-x}, 1, x, x^2, e^x \cos 3x, e^x \sin 3x$

(d) $y = x^s(d_1 \cos x + c_2 \sin x)$ $s = 1$ by fixup rule

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Name KEY Classtime ______ 2250 Midterm 3 Ver 1 [7:30, S2008]

4. **(ch6)** Complete all of the items below.

(a) [30%] Find the eigenvalues of the matrix $A = \begin{pmatrix} -2 & 7 & 1 & 27 \\ -1 & 6 & -3 & 62 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & -1 & 0 \end{pmatrix}$. To save time, **do not** find eigenvectors!

(b) [30%] Assume A is 2×2 and Fourier's model holds:

$$A\left(c_1\begin{pmatrix}1\\1\end{pmatrix} + c_2\begin{pmatrix}1\\-1\end{pmatrix}\right) = 2c_2\begin{pmatrix}1\\-1\end{pmatrix}.$$

Find A.

(c) [40%] Let $A = \begin{pmatrix} 3 & 0 & -1 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$. Circle the possible eigenvectors of A in the list below.

$$\boxed{\begin{pmatrix}-4\\2\\0\end{pmatrix}}, \boxed{\begin{pmatrix}1\\0\\0\end{pmatrix}}, \begin{pmatrix}0\\0\\1\end{pmatrix}.$$

(a) Use cofactor expansion of $\det(A-\lambda I)$ along column one to get $\det(A-\lambda I) = (-2-\lambda)(6-\lambda)\Delta + 7\Delta$ where $\Delta = \begin{vmatrix} 3-\lambda & 2 \\ -1 & -\lambda \end{vmatrix}$. Then $\Delta = 0$ or $(-2-\lambda)(6-\lambda)+7=0$. ans $= -1, 1, 2, 5$

(b) $P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $D = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$, $A = PDP^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

(c) We test $A\vec{x} = \lambda\vec{x}$ for each of the 3 vectors $\vec{x}$.

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Name KEY **Classtime** ______ 2250 Midterm 3 Ver 1 [7:30, S2008]

5. (**ch6**) Complete all parts below.

Consider the 3×3 matrix

$$E = \begin{pmatrix} 4 & 2 & -2 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix}.$$

Matrix E has a Fourier model:

$$E\left(c_1\begin{pmatrix}1\\0\\0\end{pmatrix} + c_2\begin{pmatrix}0\\1\\1\end{pmatrix} + c_3\begin{pmatrix}2\\-1\\1\end{pmatrix}\right) = 4c_1\begin{pmatrix}1\\0\\0\end{pmatrix} + 4c_2\begin{pmatrix}0\\1\\1\end{pmatrix} + 2c_3\begin{pmatrix}2\\-1\\1\end{pmatrix}.$$

(a) [40%] Find $E^3\begin{pmatrix}2\\0\\2\end{pmatrix}$ without using matrix multiply.

(b) [30%] Let $P = \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix}$, $D = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}$ and define A by $AP = PD$. Display the eigenpairs of A.

(c) [30%] Assume the vector general solution $\mathbf{x}(t)$ of the linear differential system $\mathbf{x}' = M\mathbf{x}$ is given by

$$\mathbf{x}(t) = c_1 e^{-t}\begin{pmatrix}2\\1\end{pmatrix} + c_2 e^{2t}\begin{pmatrix}-1\\1\end{pmatrix}.$$

Display an invertible matrix P and a diagonal matrix D such that $MP = PD$.

(a) $E^3\vec{x} = c_1\lambda_1^3\vec{v}_1 + c_2\lambda_2^3\vec{v}_2 + c_3\lambda_3^3\vec{v}_3$

$\vec{x} = \begin{pmatrix}2\\0\\2\end{pmatrix} = \begin{pmatrix}0\\1\\1\end{pmatrix} + \begin{pmatrix}2\\-1\\1\end{pmatrix} \Rightarrow c_1 = 0,\ c_2 = c_3 = 1$

$E^3\vec{x} = \lambda_2^3\vec{v}_2 + \lambda_3^3\vec{v}_3 = \boxed{4^3\begin{pmatrix}0\\1\\1\end{pmatrix} + 2^3\begin{pmatrix}2\\-1\\1\end{pmatrix}} = \boxed{\begin{pmatrix}16\\56\\72\end{pmatrix}}$

(b) $\left(3, \begin{pmatrix}3\\1\end{pmatrix}\right)$ and $\left(-2, \begin{pmatrix}1\\-1\end{pmatrix}\right)$

(c) $P = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$

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