Name KEY Classtime ______ 2250 Midterm 2 Ver 2 [10:45, S2008]

Applied Differential Equations 2250

Exam date: Tuesday, 11 March, 2008

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%.

1. **(Frame sequences and the 3 possibilities)**
Determine a, b such that the system has a unique solution, infinitely many solutions, or no solution:

$$\begin{aligned} 3x &+ 4by &+ 2z &= 2b \\ x &+ 2y &+ z &= 2a \\ 4x &+ 8y &+ 3z &= 2+a \end{aligned}$$

$$\left(\begin{array}{ccc|c} 3 & 4b & 2 & 2b \\ 1 & 2 & 1 & 2a \\ 4 & 8 & 3 & 2+a \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 0 & 4b-6 & -1 & 2b-6a \\ 1 & 2 & 1 & 2a \\ 4 & 8 & 3 & 2+a \end{array}\right) \text{ combo}(2,1,-3)$$

$$\left(\begin{array}{ccc|c} 0 & 4b-6 & -1 & 2b-6a \\ 1 & 2 & 1 & 2a \\ 0 & 0 & -1 & 2-7a \end{array}\right) \text{ combo}(2,3,-4)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 2a \\ 0 & 4b-6 & -1 & 2b-6a \\ 0 & 0 & -1 & 2-7a \end{array}\right) \text{ swap}(1,2)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 2a \\ 0 & 4b-6 & 0 & a+2b-2 \\ 0 & 0 & -1 & 2-7a \end{array}\right) \text{ combo}(3,2,-1)$$

answers

- If $4b-6 \neq 0$, Then 3 lead variables and a unique solution.
- If $4b-6=0$ and $a+2b-2$ [$=a+4$] is not zero, Then signal equation and no solution.
- If $4b-6=0$ and $a+4=0$, Then one free var, ∞-many solutions

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2. **(vector spaces)** Do all three parts.
(a) [20%] The vector space V is the set of all functions $f(x) = (x^2 + e^x)(a_0 + a_1 e^x + a_2 x^2 + a_3 x^7)$. Find a subspace S of V of dimension 3 which contains $e^{2x} - x^4$ and display a basis for S. Don't justify anything.

(b) [40%] Let V be the vector space of all column vectors $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and let S be the subset of V given by the equations $(x_1 + 2x_3)(x_2 + 2x_3) = 0$, $x_2 = -x_3$. Prove or disprove that S is a subspace of V.

(c) [40%] Find a basis of 4-vectors for the subspace of $\mathcal{R}^4$ given by the system of equations

$$\begin{array}{rcrcrcrcl} x_1 & + & x_2 & - & 3x_3 & + & 2x_4 & = & 0, \\ x_1 & + & 2x_2 & - & 2x_3 & + & 2x_4 & = & 0, \\ & & 2x_2 & + & 2x_3 & & & = & 0. \end{array}$$

(a) $V = \text{span}\{x^2 + e^x,\ x^2 e^x + e^{2x},\ x^4 + x^2 e^x,\ x^9 + x^7 e^x\}$

$e^{2x} - x^4 = (x^2 e^x + e^{2x}) - (x^4 + x^2 e^x) =$ l.c. of basis elements

$S = \text{span}\{x^2 + e^x,\ x^2 e^x + e^{2x},\ x^4 + x^2 e^x\}$

(b) Not a subspace. Both $\begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ are in S but their sum $= \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$ is not in S.

(c) $\begin{pmatrix} 1 & 1 & -3 & 2 \\ 1 & 2 & -2 & 2 \\ 0 & 2 & 2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -3 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -3 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -4 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\begin{cases} x_1 = 4t_1 - 2t_2 \\ x_2 = -t_1 \\ x_3 = t_1 \\ x_4 = t_2 \end{cases}$

Basis $= \left\{ \dfrac{\partial \vec{x}}{\partial t_1}, \dfrac{\partial \vec{x}}{\partial t_2} \right\} = \left\{ \begin{pmatrix} 4 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

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3. **(independence)** Do **only two** of the three parts.

(a) [50%] Let $\mathbf{u}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{u}_3 = \begin{pmatrix} 1 \\ 2 \\ 4 \\ 4 \end{pmatrix}$, State a test that decides independence or dependence of the list of three vectors [20%]. Apply the test and report the result [30%].

(b) [50%] State the pivot theorem [20%]. Then extract from the list below a largest set of independent vectors [30%].

$$\mathbf{a} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \ \mathbf{b} = \begin{pmatrix} 2 \\ -2 \\ 0 \\ 2 \end{pmatrix}, \ \mathbf{c} = \begin{pmatrix} 3 \\ -1 \\ 0 \\ 5 \end{pmatrix}, \ \mathbf{d} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \ \mathbf{e} = \begin{pmatrix} 3 \\ -2 \\ 0 \\ 4 \end{pmatrix},$$

(c) [50%] [If you did (a) and (b) already, then 100% has been attained: skip this one!]
Assume that 4×3 matrix D has nonzero orthogonal columns. Prove that there exists an invertible matrix E such that $ED = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

(a) Test: u_1, u_2, u_3 independent $\iff$ rank(aug(u_1, u_2, u_3)) = 3

$$\begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 2 \\ 1 & 1 & 4 \\ 1 & 1 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 1 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

rank = 2 $\Rightarrow$ dependent

(b) Pivot Theorem
- The pivot columns of A are independent
- The non-pivot columns of A are linear combinations of the pivot columns of A.

$$\begin{pmatrix} 1 & 2 & 3 & 0 & 3 \\ -1 & -2 & -1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 5 & 0 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & 0 & 3 \\ 0 & 0 & 2 & 0 & 1 \\ 1 & 2 & 5 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & 0 & 3 \\ 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

pivot cols = 1, 3 ans = $\{\vec{a}, \vec{c}\}$

(c) Orthogonal nonzero vectors are independent. So D has independent cols. Then rank$(D)=3$, there are 3 leading ones in rref(D). By rref$(D) = E_k \cdots E_1 D$, the result follows by taking $E = E_k \cdots E_1$.

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4. **(determinants and elementary matrices)** Do both parts.
(a) [50%] Assume given 3×3 matrices A, B. Suppose $BE_5E_4E_3 = E_2E_1A$ and E_1, E_2, E_3, E_4, E_5 are elementary matrices representing respectively a a combination, a multiply by 5, a combination, a multiply by 4 and a swap. Assume $\det(A) = 3$. Find $\det(2A(B^T)^{-1})$ [B^T is the transpose of B].
(b) [50%] Let A, B and C be 4×4 matrices such that $AB = BA$ and $C+2AB = A^2 + B^2$. Suppose C is invertible and $\mathbf{rref}(C) = E_3E_2E_1C$, where E_1, E_2, E_3 are elementary matrices representing respectively a combination, a combination and a multiply by 3. Find the possible values of $\det(A - B)$.

(a)
$$\det(2A(B^T)^{-1}) = \det(2I)\det(A)\det(B^T)^{-1}$$
$$= 8\det(A)/\det(B^T)$$
$$= 8\frac{\det(A)}{\det(B)}$$

$$\det(B)\det E_5 \det E_4 \det E_3 = \det E_2 \det E_1 \det(A)$$
$$\det(B)(-1)(4)(1) = (5)(1)\det(A)$$
$$\frac{\det(A)}{\det(B)} = \frac{(-1)(4)(1)}{(5)(1)} = \frac{-4}{5}$$
$$\det(2A(B^T)^{-1}) = 8\left(\frac{-4}{5}\right) = \frac{-32}{5}$$

(b) $C = A^2+B^2-2AB = (A-B)^2$ because $AB=BA$

$\mathrm{rref}(C) = I$ because C^{-1} exists

$I = E_3E_2E_1C \implies 1 = \det E_3 \det E_2 \det E_1 \det(C)$
$\implies 1 = (3)(1)(1)\det(C)$
$\implies \det(C) = 1/3$
$\implies \det(A-B)^2 = 1/3$
$\implies \det(A-B)\det(A-B) = \frac{1}{3}$
$\implies \det(A-B) = \pm\frac{1}{\sqrt{3}}$

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5. (inverses and Cramer's rule) Do all three parts.

(a) [20%] Determine all values of x for which A^{-1} exists: $A = \begin{pmatrix} 1 & x-1 & 1 \\ 4 & 1 & 0 \\ 2x & x & x^2 \end{pmatrix}$.

(b) [40%] Apply the adjugate formula for the inverse to find the value of the entry in row 2, column 4 of A^{-1}, given A below. Other methods are not acceptable.

$$A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 2 & 2 & 2 \\ -1 & 0 & -1 & 1 \\ 1 & 2 & 0 & 2 \end{pmatrix}$$

(c) [40%] Solve for x_2 in $A\mathbf{x} = \mathbf{b}$ by Cramer's rule. Other methods are not acceptable.

$$A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 3 & 0 & 4 & 1 \\ 5 & 6 & 8 & 4 \\ 1 & 0 & 1 & 0 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}.$$

(a) $\det(A) = \begin{vmatrix} 1 & x-1 & 1 \\ 4 & 1 & 0 \\ 2x & x & x^2 \end{vmatrix} = (1)(+1)\begin{vmatrix} 4 & 1 \\ 2x & x \end{vmatrix} + x^2(+1)\begin{vmatrix} 1 & x-1 \\ 4 & 1 \end{vmatrix}$

$= (4x - 2x) + x^2(1 - 4x + 4)$

$= 2x + 5x^2 - 4x^3$

A^{-1} exists $\Leftrightarrow \det(A) \neq 0 \Leftrightarrow x \neq 0, \frac{5 \pm \sqrt{57}}{8}$

(b) $A^{-1}[2,4] = \dfrac{\mathrm{cof}(A,4,2)}{\det(A)}$ $\quad$ $\mathrm{Cof} = \begin{vmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ -1 & -1 & 1 \end{vmatrix}(-1)^{4+2} = 4$

$= \dfrac{4}{-2} = \boxed{-2}$ $\quad$ $\det = -2$ $\quad$ use combo(1,4,-1) to reduce to 3x3 minor(A, 4,4)

(c) $x_2 = \dfrac{\Delta_2}{\Delta}$ $\quad$ $\Delta = \begin{vmatrix} 1 & 2 & 0 & 0 \\ 3 & 0 & 4 & 1 \\ 5 & 6 & 8 & 4 \\ 1 & 0 & 1 & 0 \end{vmatrix} = 4$

$= \dfrac{-2}{4}$ $\quad$ $\Delta_2 = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 3 & 0 & 4 & 1 \\ 5 & -1 & 8 & 4 \\ 1 & 0 & 1 & 0 \end{vmatrix} = (1)(-1)(-1)(0-1) + (1)(-1)\begin{vmatrix} 3 & 4 & 1 \\ 5 & 8 & 4 \\ 1 & 1 & 0 \end{vmatrix} = -2$

$= \boxed{\dfrac{-1}{2}}$

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