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Applied Differential Equations 2250

Exam date: Tuesday, 11 March, 2008

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%.

1. **(Frame sequences and the 3 possibilities)**
Determine a, b such that the system has a unique solution, infinitely many solutions, or no solution:

$$\begin{array}{rcrcrcl} x &+& 2y &+& z &=& 2a \\ 3x &+& 2by &+& 2z &=& b \\ 4x &+& 8y &+& 3z &=& 2+a \end{array}$$

$$A = \begin{pmatrix} 1 & 2 & 1 & 2a \\ 3 & 2b & 2 & b \\ 4 & 8 & 3 & 2+a \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 1 & 2a \\ 0 & 2b-6 & -1 & b-6a \\ 4 & 8 & 3 & 2+a \end{pmatrix} \text{combo}(1,2,-3)$$

$$\begin{pmatrix} 1 & 2 & 1 & 2a \\ 0 & 2b-6 & -1 & b-6a \\ 0 & 0 & -1 & 2-7a \end{pmatrix} \text{combo}(1,3,-4)$$

$$\begin{pmatrix} 1 & 2 & 1 & 2a \\ 0 & 2b-6 & 0 & b+a-2 \\ 0 & 0 & -1 & 2-7a \end{pmatrix} \text{combo}(3,2,-1)$$

- If $b \neq 3$, Then 3 lead vars and unique sol
- If $b = 3$ and $a+1 \neq 0$, Then signal eq and No sol.
- If $b = 3$ and $a+1 = 0$, Then one free var and ∞-many sols

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2. (vector spaces) Do all three parts.

(a) [20%] The vector space V is the set of all polynomials $p(x) = (1+x^2)(a_0 + a_1 x + a_2 x^5 + a_3 x^7)$. Find a subspace S of V of dimension 3 which contains $x^5 - x^9$ and display a basis for S. Don't justify anything.

(b) [40%] Let V be the vector space of all column vectors $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and let S be the subset of V given by the equations $x_1 + 2x_3 = 0$, $x_2 + x_3 = 0$, $x_2 = -x_3$. Prove or disprove that S is a subspace of V.

(c) [40%] Find a basis of 4-vectors for the subspace of $\mathcal{R}^4$ given by the system of equations

$$\begin{array}{ccccccccc} x_1 & + & x_2 & - & 3x_3 & + & 3x_4 & = & 0, \\ x_1 & + & 2x_2 & - & 2x_3 & + & 3x_4 & = & 0, \\ & & 2x_2 & + & 2x_3 & & & = & 0. \end{array}$$

(a) Basis for $V = \{1+x^2,\ x+x^3,\ x^5+x^7,\ x^7+x^9\}$

Because $x^5 - x^9 = (x^5+x^7) - (x^7+x^9)$ = l.c. of basis elements

Then $S = \text{span}\{1+x^2,\ x^5+x^7,\ x^7+x^9\}$ has dim = 3 and contains x^5-x^9

(b) Define $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$. Then $S = \{\vec{v} : A\vec{v} = \vec{0}\}$ is a subspace of V by the kernel theorem.

(c) $\begin{pmatrix} 1 & 1 & -3 & 3 \\ 1 & 2 & -2 & 3 \\ 0 & 2 & 2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -3 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -3 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\to \begin{pmatrix} 1 & 0 & -4 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $\left\{\begin{array}{l} x_1 - 4x_3 + 3x_4 = 0 \\ x_2 + x_3 = 0 \\ 0 = 0 \end{array}\right.$ $\left\{\begin{array}{l} x_1 = 4t_1 - 3t_2 \\ x_2 = -t_1 \\ x_3 = t_1 \\ x_4 = t_2 \end{array}\right.$

Basis = $\left\{ \dfrac{\partial \vec{x}}{\partial t_1}, \dfrac{\partial \vec{x}}{\partial t_2} \right\}$

Basis = $\left\{ \begin{pmatrix} 4 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

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3. **(independence)** Do **only two** of the three parts.

(a) [50%] Let $\mathbf{u}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{u}_3 = \begin{pmatrix} 1 \\ 1 \\ 3 \\ 3 \end{pmatrix}$, State a test that decides independence or dependence of the list of three vectors [20%]. Apply the stated test and report the result [30%].

(b) [50%] State the pivot theorem [20%]. Then extract from the list below a largest set of independent vectors [30%].

$$\mathbf{a} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 2 \\ -2 \\ 0 \\ 2 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \mathbf{d} = \begin{pmatrix} 3 \\ -1 \\ 0 \\ 5 \end{pmatrix}, \mathbf{e} = \begin{pmatrix} 3 \\ -2 \\ 0 \\ 4 \end{pmatrix},$$

(c) [50%] [If you did (a) and (b) already, then 100% has been attained: skip this one!]
Assume that 4×3 matrix D has exactly one non-pivot column. Prove (or disprove) that there exists an invertible matrix E such that $ED = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

(a) u_1, u_2, u_3 are independent $\iff$ rank(aug (u_1, u_2, u_3)) = 3

$$A = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & 3 \\ 1 & 1 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 3 \\ 1 & 1 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

rank(A) = 2 $\Rightarrow$ dependent

(b) Pivot Theorem
- The pivot columns of A are independent
- The non-pivot columns of A of linear combinations of the pivot columns of A.

$$A = \begin{pmatrix} 1 & 2 & 0 & 3 & 3 \\ -1 & -2 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 5 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 0 & 3 & 3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 5 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 0 & 3 & 3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 2 & 0 & 3 & 3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$ Pivot Cols = 1, 4 ans = $\{\vec{a}, \vec{d}\}$

(c) Then D has 2 pivot columns. The result is false, because $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ has exactly one non-pivot col and rref(D) = D.

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4. **(determinants and elementary matrices)** Do both parts.

(a) [50%] Assume given 3×3 matrices A, B. Suppose $BE_5E_4E_3 = E_2E_1A$ and E_1, E_2, E_3, E_4, E_5 are elementary matrices representing respectively a a combination, a multiply by 3, a swap, a multiply by 4 and a swap. Assume $\det(A) = 3$. Find $\det(2A(B^T)^{-1})$ [B^T is the transpose of B].

(b) [50%] Let A, B and C be 4×4 matrices such that $AB = BA$ and $C + 4AB = A^2 + 4B^2$. Suppose C is invertible and $\mathbf{rref}(C) = E_2E_1C$, where E_1, E_2 are elementary combination matrices. Find the possible values of $\det(A - 2B)$.

(a)

$$\begin{aligned}\det(2A(B^T)^{-1}) &= \det(2I)\det(A)\det(B^T)^{-1} \\ &= 8\det(A)\det(B^{-1})^T \\ &= 8\det(A)\det B^{-1} \\ &= 8\frac{\det(A)}{\det(B)}\end{aligned}$$

$\det(B)\det E_5 \det E_4 \det E_3 = \det E_2 \det E_1 \det(A)$ by product rule

$\det(B)(-1)(4)(-1) = (3)\cdot(1)\det(A)$

$$\frac{\det(A)}{\det(B)} = \frac{(-1)(4)(-1)}{(3)(1)} = \frac{4}{3}$$

$$\det(2A(B^T)^{-1}) = 8\left(\frac{4}{3}\right) = \frac{32}{3}$$

(b)

$C = A^2 + 4B^2 - 4AB$

$C = (A-2B)(A-2B)$ because $AB = BA$

$\det C = \det(A-2B)\det(A-2B)$

$$\det C = \frac{\det(\mathrm{rref}(C))}{\det E_2 \det E_1} = \frac{1}{1 \cdot 1}$$ because $\mathrm{rref}(C) = I$

$\det(A-2B)^2 = 1$

$\det(A-2B) = \pm 1$

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5. **(inverses and Cramer's rule)** Do all three parts.

(a) [20%] Determine all values of x for which A^{-1} exists: $A = \begin{pmatrix} 1 & x-1 & 1 \\ 3 & 2 & 0 \\ 2x & x & x^2 \end{pmatrix}$.

(b) [40%] Apply the adjugate formula for the inverse to find the value of the entry in row 3, column 4 of A^{-1}, given A below. Other methods are not acceptable.

$$A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 2 & 2 & 2 \\ -1 & 0 & -1 & 1 \\ 1 & 2 & 0 & 2 \end{pmatrix}$$

(c) [40%] Solve for x_3 in $A\mathbf{x} = \mathbf{b}$ by Cramer's rule. Other methods are not acceptable.

$$A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 5 & 6 & 8 & 4 \\ 3 & 0 & 4 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 2 \\ -2 \\ 0 \\ 0 \end{pmatrix}.$$

(a)
$$\det(A) = 1(+1)\begin{vmatrix} 3 & 2 \\ 2x & x \end{vmatrix} + x^2(+1)\begin{vmatrix} 1 & x-1 \\ 3 & 2 \end{vmatrix}$$
$$= -x + x^2(2 - 3x + 3)$$
$$= -3x^3 + 5x^2 - x$$
$$= -x(3x^2 - 5x + 1)$$

A^{-1} exists for $x \neq 0$ and $3x^2 - 5x + 1 \neq 0$ $\left[x \neq \frac{5 \pm \sqrt{13}}{6}\right]$

(b) $B = A^{-1}$.

$$B[3,4] = \frac{\text{cofactor}(A,4,3)}{\det(A)}$$

$$\text{Cofactor} = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 2 & 2 \\ -1 & 0 & 1 \end{vmatrix}(-1)^{4+3} = (-4)(-1) = 4$$

$\det(A) = -2$

$$\boxed{B[3,4] = -2}$$

(c) $x_3 = \Delta_3/\Delta$

$$\Delta = \begin{vmatrix} 1 & 2 & 0 & 0 \\ 5 & 6 & 8 & 4 \\ 3 & 0 & 4 & 1 \\ 1 & 0 & 1 & 0 \end{vmatrix} = -4$$

$$\Delta_3 = \begin{vmatrix} 1 & 2 & 2 & 0 \\ 5 & 6 & -2 & 4 \\ 3 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{vmatrix} = (-1)(-4-12) = 16$$

$x_3 = -4$

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