Applied Differential Equations 2250

Exam date: Tuesday, 11 March, 2008

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%.

1. (Frame sequences and the 3 possibilities)

Determine $a$, $b$ such that the system has a unique solution, infinitely many solutions, or no solution:

$$
\begin{align*}
x + 2y + z &= 2a \\
3x + 2by + 2z &= b \\
4x + 8y + 3z &= 2 + a
\end{align*}
$$

$$
A = \begin{pmatrix}
1 & 2 & 1 & 2a \\
3 & 2b & 2 & b \\
4 & 8 & 3 & 2+a
\end{pmatrix}
$$

\[ \text{combo(1,2,3)} \]

$$
\begin{pmatrix}
1 & 2 & 1 & 2a \\
0 & 2b-6 & -1 & b-6a \\
0 & 0 & 3 & 2+a
\end{pmatrix}
$$

\[ \text{combo(1,3,4)} \]

$$
\begin{pmatrix}
1 & 2 & 1 & 2a \\
0 & 2b-6 & 0 & b+2a \\
0 & 0 & -1 & 2-7a
\end{pmatrix}
$$

\[ \text{combo(3,2,-1)} \]

- If $b \neq 3$, then 3 lead vars and unique sol
- If $b = 3$ and $a+1 \neq 0$, then signed eq and no sol.
- If $b = 3$ and $a+1 = 0$, then one free var and $\infty$ many sols

Use this page to start your solution. Attach extra pages as needed, then staple.
2. (vector spaces) Do all three parts.
   (a) [20%] The vector space $V$ is the set of all polynomials $p(x) = (1 + x^2)(a_0 + a_1 x + a_2 x^5 + a_3 x^7)$.
   Find a subspace $S$ of $V$ of dimension 3 which contains $x^5 - x^9$ and display a basis for $S$. Don't justify anything.

   (b) [40%] Let $V$ be the vector space of all column vectors \[
   \begin{pmatrix}
   x_1 \\
   x_2 \\
   x_3 
   \end{pmatrix}
   \]
   and let $S$ be the subset of $V$ given by the equations $x_1 + 2x_3 = 0$, $x_2 + x_3 = 0$, $x_2 = -x_3$. Prove or disprove that $S$ is a subspace of $V$.

   (c) [40%] Find a basis of 4-vectors for the subspace of $\mathbb{R}^4$ given by the system of equations

   \[
   \begin{align*}
   x_1 + x_2 - 3x_3 + 3x_4 &= 0, \\
   x_1 + 2x_2 - 2x_3 + x_4 &= 0, \\
   2x_2 + 2x_3 &= 0.
   \end{align*}
   \]

   \[\text{(a) Basis for } V = \{1 + x^2, x + x^3, x^5 + x^7, x^7 + x^9\}\]

   Because $x^5 - x^9 = (x^5 + x^7) - (x^7 + x^9) = \text{l.c. of basis elements}$

   Then $S = \text{span} \{1 + x^2, x^5 + x^7, x^7 + x^9\}$ has dim = 3 and contains $x^5 - x^9$.

   \[\text{(b) Define } A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix}. \text{ Then } S = \{ \vec{v} : A\vec{v} = \vec{0}\} \text{ is a subspace of } V \text{ by the Kernel Theorem.}\]

   \[\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 0 \\ 2 & 2 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \]

   \[\rightarrow \begin{pmatrix} 1 & 0 & -4 & 3 \\ 0 & 1 & 1 & 0 \end{pmatrix} \begin{cases} x_1 - 4x_3 + 3x_4 = 0 \\ x_2 + x_3 = 0 \end{cases} \begin{pmatrix} x_1 = 4t_1 - 3t_2 \\ x_2 = -t_1 \\ x_3 = t_1 \\ x_4 = t_2 \end{pmatrix}\]

   Basis = \{ \begin{pmatrix} 4 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 0 \end{pmatrix} \}

   Basis = \{ \begin{pmatrix} \frac{\partial x}{\partial t_1} \\ \frac{\partial x}{\partial t_2} \end{pmatrix} \}

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3. (independence) Do only two of the three parts.

(a) [50%] Let \( u_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 1 \\ 1 \\ 3 \\ 3 \end{pmatrix} \). State a test that decides independence or dependence of the list of three vectors [20%]. Apply the stated test and report the result [30%].

(b) [50%] State the pivot theorem [20%]. Then extract from the list below a largest set of independent vectors [30%].

\[
a = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ -2 \\ 0 \\ 2 \end{pmatrix}, \quad c = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad d = \begin{pmatrix} 3 \\ -1 \\ 0 \\ 5 \end{pmatrix}, \quad e = \begin{pmatrix} 3 \\ -2 \\ 0 \\ 4 \end{pmatrix},
\]

(c) [50%] [If you did (a) and (b) already, then 100% has been attained: skip this one!]

Assume that a \( 4 \times 3 \) matrix \( D \) has exactly one non-pivot column. Prove that there exists an invertible or disprove matrix \( E \) such that \( ED = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \).

\[ \begin{align*}
\text{(a)} & \quad u_1, u_2, u_3 \text{ are independent } \iff \text{rank}(\text{any } (u_1, u_2, u_3)) = 3 \\
A &= \begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & 3 \\ 0 & 1 & 2 \end{pmatrix} \\
\text{rank}(A) = 2 & \Rightarrow \text{dependent}
\end{align*} \]

\[ \begin{align*}
\text{(b)} & \quad \text{Pivot Theorem} \\
& \quad \text{The pivot columns of } A \text{ are independent} \\
& \quad \text{The non-pivot columns of } A \text{ of linear combinations of the pivot columns of } A, \\
A &= \begin{pmatrix} -1 & -2 & 0 & 3 \\ 1 & 2 & 0 & 3 \\ 1 & 0 & 5 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
& \Rightarrow \begin{pmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 2 \\ 1 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
& \Rightarrow \begin{pmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
& \text{Pivot cols} = 1, 4 \quad \text{Ens} = \{a, d\}
\end{align*} \]

\[ \begin{align*}
\text{(c)} & \quad \text{Then } D \text{ has 2 pivot columns. The result is false, because} \\
D &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \text{ has exactly one non-pivot col and } \text{ref}(D) = D.
\end{align*} \]
4. (determinants and elementary matrices) Do both parts.
   (a) [50%] Assume given $3 \times 3$ matrices $A$, $B$. Suppose $BE_5E_4E_3 = E_2E_1A$ and $E_1$, $E_2$, $E_3$, $E_4$, $E_5$ are elementary matrices representing respectively a combination, a multiply by 3, a swap, a multiply by 4 and a swap. Assume $\det(A) = 3$. Find $\det(2A(B^T)^{-1})$ [$B^T$ is the transpose of $B$].

   (b) [50%] Let $A$, $B$ and $C$ be $4 \times 4$ matrices such that $AB = BA$ and $C + 4AB = A^2 + 4B^2$. Suppose $C$ is invertible and $\text{rref}(C) = E_2E_1C$, where $E_1$, $E_2$ are elementary combination matrices. Find the possible values of $\det(A - 2B)$.

\[ \det(2A(B^T)^{-1}) = \frac{\det(2I) \det(A) \det(B^T)^{-1}}{\det(B)} = \frac{8 \det(A) \det(B^{-1})^T}{\det(B)} = \frac{8 \det(A) \det(B^{-1})}{\det(B)} \]

\[ \det(B) \det(E_5) \det(E_4) \det(E_3) = \det(E_2) \det(E_1) \det(A) \quad \text{by product rule} \]

\[ \frac{\det(A)}{\det(B)} = \frac{(-1)(4)(-1)}{(3)(1)} = \frac{4}{3} \]

\[ \det(2A(B^T)^{-1}) = 8 \left( \frac{4}{3} \right) = \frac{32}{3} \]

\[ C = A^2 + 4B^2 - 4AB \quad \text{because } AB = BA \]

\[ \det C = \det(A - 2B) \det(A - 2B) \]

\[ \det C = \frac{\det(\text{rref}(C))}{\det(E_2) \det(E_1)} = \frac{1}{1 \cdot 1} \quad \text{because } \text{rref}(C) = I \]

\[ \det(A - 2B)^2 = 1 \]

\[ \det(A - 2B) = \pm 1 \]

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5. (inverses and Cramer’s rule) Do all three parts.

(a) [20%] Determine all values of \( x \) for which \( A^{-1} \) exists: \( A = \begin{pmatrix} 1 & x - 1 & 1 \\ 3 & 2 & 0 \\ 2x & x & x^2 \end{pmatrix} \).

(b) [40%] Apply the adjugate formula for the inverse to find the value of the entry in row 3, column 4 of \( A^{-1} \), given \( A \) below. Other methods are not acceptable.

\[
A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 2 & 2 & 2 \\ -1 & 0 & -1 & 1 \\ 1 & 2 & 0 & 2 \end{pmatrix}
\]

(c) [40%] Solve for \( x_3 \) in \( Ax = b \) by Cramer’s rule. Other methods are not acceptable.

\[
A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 5 & 6 & 8 & 4 \\ 3 & 0 & 4 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ -2 \\ 0 \\ 0 \end{pmatrix}
\]

\[\text{det}(A) = 1 \begin{vmatrix} 3 & 2 \\ 2x & x \end{vmatrix} + x^2(1+1) \begin{vmatrix} 1 & 3 \\ -1 & 2 \end{vmatrix} = -x + x^2(2 - 3x + 3) = -3x^3 + 5x^2 - x = -x(3x^2 - 5x + 1)\]

\[A^{-1} \text{ exists for } x \neq 0 \text{ and } 3x^2 - 5x + 1 \neq 0 \]

\[x_3 = \frac{\text{cofactor}(A,4,2)}{\text{det}(A)} = \frac{1}{-2} = -\frac{1}{2}
\]

\[B = A^{-1}, \quad B[3,4] = \frac{\text{cofactor}(A,3,4)}{\text{det}(A)} = \frac{1}{-2} = -\frac{1}{2}\]

\[x_3 = \frac{\Delta_2}{\Delta} = \frac{1}{-2} = -\frac{1}{2}
\]

\[\Delta = \begin{vmatrix} 1 & 2 & 0 & 0 \\ 5 & 6 & 8 & 4 \\ 3 & 0 & 4 & 1 \\ 1 & 0 & 1 & 0 \end{vmatrix} = -4
\]

\[\Delta_3 = \begin{vmatrix} 1 & 2 & 2 & 0 \\ 5 & 6 & 2 & 4 \\ 3 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{vmatrix} = (-1)(-4-12) = 16
\]

\[x_3 = -4
\]