Name KEY Time of your class ______

Differential Equations and Linear Algebra 2250

Midterm Exam 1 Version 2 [10:45]
Tuesday, 12 February 2008

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

1. **(Quadrature Equations)**
   (a) [25%] Solve $y' = \dfrac{1+x^2}{1-x^2}$.
   (b) [25%] Solve $y' = \sec x + \tan x$.
   (c) [25%] Solve $y' = \dfrac{\sin(\ln|x|)}{x}$, $y(1) = 3$.
   (d) [25%] Find the position $x(t)$ from the velocity model $\dfrac{d}{dt}(e^{-t}\,v) = 2e^t$ and the position model $\dfrac{dx}{dt} = v(t)$.

(a) $y = \int \frac{1+x^2}{1-x^2}dx = -x - \ln|x-1| + \ln|x+1| + c$

Detail: $\frac{1+x^2}{1-x^2} = -1 + \frac{-2}{x^2-1} = -1 + \frac{1}{x+1} - \frac{1}{x-1}$

(b) $y = \int(\sec x + \tan x)dx = \ln|\sec x + \tan x| + \ln|\sec x| + C$

Detail: std integral table entries
Also: $-\ln|\cos x| = \ln|\sec x|$

(c) $y = \int \sin u \, du \qquad u = \ln|x|$

$y = -\cos(\ln|x|) + C \qquad C = 4$ because $3 = -\cos 0 + C$

$\boxed{y = 4 - \cos(\ln|x|)}$

(d) $\int \frac{d}{dt}(e^{-t}v)dt = \int 2e^t$

$e^{-t}v = 2e^t + c_1$

$v = 2e^{2t} + c_1 e^t$

$x' = 2e^{2t} + c_1 e^t$

$\boxed{x = \frac{2}{3}e^{2t} + c_1 e^t + c_2}$

Use this page to start your solution. Attach extra pages as needed, then staple.

Name. KEy

2. (Classification of Equations)
The differential equation $y' = f(x,y)$ is defined to be **separable** provided $f(x,y) = F(x)G(y)$ for some functions F and G.

(a) [40%] Check ($\boxed{\text{X}}$) the problems that can be put into separable form, but don't supply any details.

$\boxed{\text{X}}$	$y' = y(2x+3) + xy + 2y$	$\boxed{}$	$y' = 1+(x-1)(y+1)-xy$
$\boxed{\text{X}}$	$y' = 2e^{2x}e^{y} + xe^{2x+y}$	$\boxed{}$	$y' + \tan y = x$

(b) [10%] State a calculus test which can verify that an equation $y' = f(x,y)$ is linear but not quadrature.

(c) [10%] Give an example of $y' = f(x,y)$ which is separable but not quadrature and not linear. No details expected.

(d) [40%] Apply a separable equation test to show that $y' = e^x + xe^y$ is not separable.

(a)
- $y' = (2x+3+x+2)y$
- $y' = (2e^{2x}+xe^{2x})e^{y}$
- $y' = 1+xy-y+x-1-xy = -y+x$ Not sep
- $y' = x - \tan y$ Not Sep

(b) $\frac{\partial f}{\partial y}$ indep of y but not 0

(c) $y' = x^2+y^2$ $\qquad \frac{f_y}{f} = \frac{2y}{x^2+y^2}$ depends on $x \Rightarrow$ Not Sep.

(d)
$$f = e^x + xe^y$$
$$f_y = xe^y$$
$$\frac{f_y}{f} = \frac{xe^y}{e^x+xe^y} = \frac{1}{\frac{e^x}{x}e^{-y}+1}$$
depends on $x \Rightarrow$ Not Sep.

Name. KEY

2250 Exam 1 Ver 2 S2008 [10:45]
Time of your class ______

3. **(Solve a Separable Equation)**
Given $yy' = \left(\dfrac{\cos^2 x}{\tan x} + \dfrac{2x^2+6}{2+x}\right)(1-y)(2+y)$.

Find a non-equilibrium solution in implicit form.
To save time, **do not solve** for y explicitly and **do not solve** for equilibrium solutions.

$$\frac{y y'}{(1-y)(2+y)} = \frac{\cos^3 x}{\sin x} + \frac{2x^2+6}{2+x}$$

$$\frac{-1/3}{y-1} + \frac{-2/3}{y+2} = \frac{(1-\sin^2 x)\cos x}{\sin x} + 2x - 4 + \frac{14}{x+2}$$

pythagorean identity $\cos^2\theta + \sin^2\theta = 1$

$$= \cot x - \sin x \cos x + 2x - 4 + \frac{14}{x+2}$$

Quadrature step

$$\boxed{-\frac{1}{3}\ln|y-1| - \frac{2}{3}\ln|y+2| = \ln|\sin x| + \frac{1}{2}\cos^2(x) + x^2 - 4x + 14\ln|x+2| + c}$$

Details

$$\begin{array}{r} 2x - 4 \\ x+2 \overline{\big) 2x^2 + 6} \\ \underline{2x^2 + 4x} \\ -4x + 6 \\ \underline{-4x - 8} \\ 14 \end{array} \xrightarrow{\text{Div. alg.}} \frac{2x^2+6}{x+2} = 2x - 4 + \frac{14}{x+2}$$

$$\frac{y}{(1-y)(2+y)} = \frac{A}{y-1} + \frac{B}{y+2}$$

$$-y = A(y+2) + B(y-1)$$

clear fractions

$y=1: \quad -1 = 3A + 0$

$y=-2: \quad 2 = 0 - 3B$

$\longrightarrow \boxed{A = -1/3, \quad B = -2/3}$

Heaviside's coverup also works quickly

Use this page to start your solution. Attach extra pages as needed, then staple.

Name. KEY

2250 Exam 1 Ver 2 S2008 [10:45]
Time of your class ______

4. **(Linear Equations)**

(a) [60%] Solve the linear model $10x'(t) = -70 + \frac{10}{2t+5}x(t)$, $x(0) = -35$. Show all integrating factor steps.

(b) [20%] Solve the homogeneous equation $3\frac{dy}{dx} = -(4x^3)y$.

(c) [20%] Solve $\frac{dy}{dx} = -2y + 3$ using the superposition principle $y = y_h + y_p$. Expected are answers for y_h and y_p.

(a) $x' - \frac{1}{2t+5}x = -7$

$W = e^{\int p\,dt}$

$W = e^{-\int \frac{dt}{2t+5}}$

$W = (2t+5)^{-1/2}$

Best factor because W is defined near t = 0.

$\frac{(Wx)'}{W} = -7$

$(Wx)' = -7W$

$Wx = -7\int (2t+5)^{-1/2}\,dt$

$= -7\frac{(2t+5)^{1/2}}{1/2}\frac{1}{2} + c$

$x = -7(2t+5) + c(2t+5)^{1/2}$

$c = 0$ because $-35 = -7(0+5) + c\sqrt{5}$

$\boxed{x = -14t - 35}$

(b) $y' = -\frac{4}{3}x^3 y$

$y' + \frac{4}{3}x^3 y = 0$

$W = e^{\int \frac{4}{3}x^3\,dx}$

$W = e^{x^4/3}$

$(Wy)'/W = 0$

$Wy = c \implies \boxed{y = c e^{-x^4/3}}$

(c) $\boxed{\begin{array}{l} y_p = 3/2 \\ y_h = ce^{-2x} \end{array}}$

Equilibrium Solution

Growth Decay sol'n $y' = ky$ is $y = ce^{kx}$

Use this page to start your solution. Attach extra pages as needed, then staple.

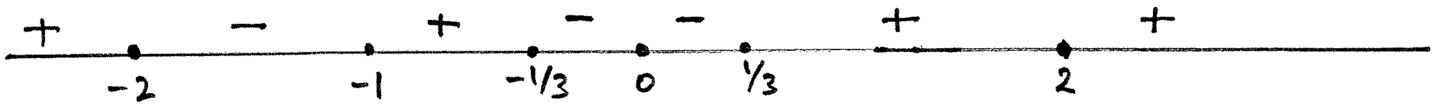
Name. KEY

2250 Exam 1 Ver 2 S2008 [10:45]
Time of your class ______

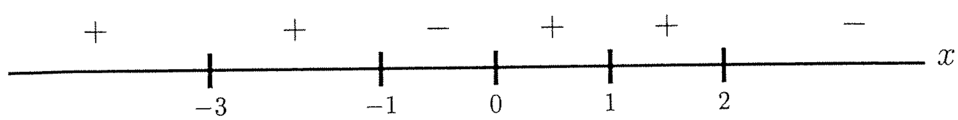
5. **(Stability)**
(a) [50%] Draw a phase line diagram for the differential equation

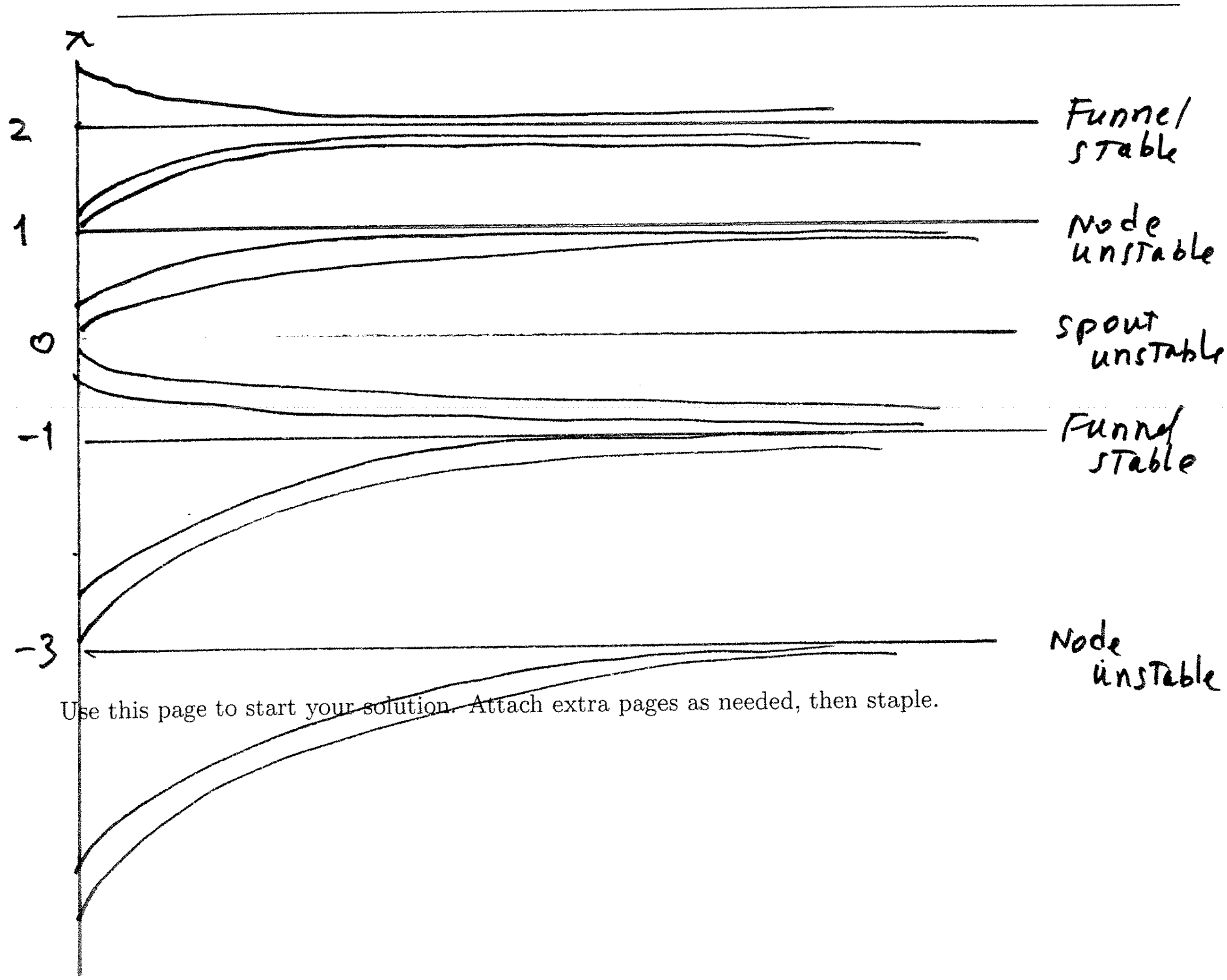
$$\frac{dx}{dt} = \ln(1+3x^2)\left(1-\sqrt[4]{|3x|}\right)^3 (1+x)(4-x^2)(x-2)^3.$$

Expected in the phase line diagram are equilibrium points and signs of dx/dt.



(b) [50%] Draw a phase diagram with at least 10 threaded curves, using the phase line diagram given below. Add these labels as appropriate: funnel, spout, neither spout nor funnel [a node], stable, unstable. A direction field is not expected nor required.





Use this page to start your solution. Attach extra pages as needed, then staple.