Name KEY Time of your class ______

Differential Equations and Linear Algebra 2250

Midterm Exam 1 Version 1 [7:30]
Tuesday, 12 February 2008

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

1. **(Quadrature Equations)**
   (a) [25%] Solve $y' = \dfrac{1-x^2}{1+x^2}$.
   (b) [25%] Solve $y' = (\sec x + \tan x)^2$.
   (c) [25%] Solve $y' = \dfrac{\cos(\ln|x|)}{x}$, $y(1) = 2$.
   (d) [25%] Find the position $x(t)$ from the velocity model $\dfrac{d}{dt}(e^t v) = e^{-t}$ and the position model $\dfrac{dx}{dt} = v(t)$.

(a) $y = \int \frac{1-x^2}{1+x^2}dx = \int\left(-1 + \frac{2}{1+x^2}\right)dx = -x + \tan^{-1}(x) + c$

(b) $y = \int(\sec^2 x + 2\sec x\tan x + \tan^2 x)\,dx$

$= \int(2\sec^2 x - 1 + 2\sec x \tan x)\,dx$

$= 2\tan(x) - x + 2\sec x + c$

$1 + \tan^2\theta = \sec^2\theta$

(c) $y = \int \cos u\,du$, $u = \ln|x|$

$= \sin u + c = \sin(\ln|x|) + c$

$= \sin(\ln|x|) + 2$

$\ln|1| = 0$
$\sin 0 = 0$
Because $c = 2$

(d) $\int \frac{d}{dt}(e^t v)\,dt = \int e^{-t}dt$

$e^t v = -e^{-t} + c_1$

$v = -e^{-2t} + c_1 e^{-t}$

$x' = -e^{-2t} + c_1 e^{-t}$

$x = \frac{1}{2}e^{-2t} + c_2 e^{-t} + c_3$

absorb $-$ into c_2

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Name. KEY

2250 Exam 1 Ver 1 S2008 [7:30]
Time of your class ______

2. **(Classification of Equations)**
The differential equation $y' = f(x, y)$ is defined to be **separable** provided $f(x, y) = F(x)G(y)$ for some functions F and G.

(a) [40%] Check ($\boxed{X}$) the problems that can be put into separable form, but don't supply any details.

$\boxed{X}$ $y' = y(2x+3) + (x-2)y$ $= 2xy + 3y + xy - 2y$	$\square$ $y' = (x-1)(y+1) - xy$
$\boxed{X}$ $y' = 2e^{2x}e^{2y} + e^{2x+y}$	$\boxed{X}$ $y' + \tan y = 1$

(b) [10%] State a calculus test which can verify that an equation $y' = f(x, y)$ is linear.

(c) [10%] Give an example of $y' = f(x, y)$ which is linear but not quadrature and not separable. No details expected.

(d) [40%] Apply a separable equation test to show that $y' = e^x + e^y$ is not separable.

(a) • $f = (3x+1)y$ • $f = x - y - 1$, $\frac{f_y}{f}$ depends on x

• $f = e^{2x}(2e^{2y} + e^y)$ • $f = 1 - \tan y$

(b) $\frac{\partial f}{\partial y}$ independent of $y \iff y' = f(x,y)$ is linear

(c) $y' + y = x$

$f = x - y$ $\quad f_y = -1$ indep of $y \Rightarrow$ Linear

$f_y \neq 0 \Rightarrow$ not quadrature

$\frac{f_y}{f} = \frac{-1}{x-y}$ depends on $x \Rightarrow$ Not separable

(d) Let $f = e^x + e^y$

Then $\frac{f_y}{f} = \frac{e^y}{e^x + e^y}$ depends on $x \Rightarrow y' = f(x,y)$ not separable.

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Name. KEY

2250 Exam 1 Ver 1 S2008 [7:30]
Time of your class ______

3. (Solve a Separable Equation)
Given $yy' = \left(\dfrac{\sin^2 x}{\cot x} + \dfrac{2x^2+6}{3+x}\right)(y+1)(2-y)$.

Find a non-equilibrium solution in implicit form.
To save time, **do not solve** for y explicitly and **do not solve** for equilibrium solutions.

$$\frac{-yy'}{(y+1)(y-2)} = \sin^3 x/\cos x + 2x - 6 + \frac{24}{x+3}$$

$$\frac{-1/3}{y+1} + \frac{-2/3}{y-2} = (1-\cos^2 x)\frac{\sin x}{\cos x} + 2x - 6 + \frac{24}{x+3} \qquad u = \sin(x)$$

Quadrature step

$$-\frac{1}{3}\ln|1+y| - \frac{2}{3}\ln|y-2| = \ln|\sec x| - \frac{\sin^2 x}{2} + x^2 - 6x + 24\ln|x+3| + C$$

Details

$$\begin{array}{r|l} & 2x-6 \\ \hline x+3 & 2x^2+6 \\ & \underline{2x^2+6x} \\ & -6x+6 \\ & \underline{-6x-18} \\ & 24 \end{array} \quad \xrightarrow{\text{Div. alg.}} \quad \frac{2x^2+6}{x+3} = 2x - 6 + \frac{24}{x+3}$$

$$\frac{-y}{(y+1)(y-2)} = \frac{A}{y+1} + \frac{B}{y-2}$$

$$-y = A(y-2) + B(y+1) \qquad \text{clear fractions}$$

$$\begin{aligned} y=-1:&\quad 1 = -3A + 0 \\ y=2:&\quad -2 = 0 + 3B \end{aligned} \quad \longrightarrow \quad A = \frac{-1}{3},\ B = \frac{-2}{3}$$

Heaviside's coverup also acceptable.

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Name. KEY

2250 Exam 1 Ver 1 S2008 [7:30]
Time of your class ______

4. **(Linear Equations)**

(a) [60%] Solve the linear model $10x'(t) = -110 + \frac{10}{2t+5}x(t)$, $x(0) = -55$. Show all integrating factor steps.

(b) [20%] Solve the homogeneous equation $3\frac{dy}{dx} = -(2x^2)y$.

(c) [20%] Solve $\frac{dy}{dx} = 5y + 2$ using the superposition principle $y = y_h + y_p$. Expected are answers for y_h and y_p.

(a)

$$x' - \frac{1}{2t+5}x = -11$$

$$W = e^{\int p\,dt}$$
$$W = e^{-\frac{1}{2}\ln|2t+5|}$$
$$W = (2t+5)^{-1/2}$$

best factor because W defined at t = 0.

$$\frac{(Wx)'}{W} = -11$$
$$(Wx)' = -11\,W$$
$$Wx = -11\int (2t+5)^{-1/2}dt$$
$$= -11\,\frac{(2t+5)^{1/2}}{1/2}\,\frac{1}{2} + c$$
$$x = -11(2t+5) + c(2t+5)^{1/2}$$
$$-55 = -11(0+5) + c\sqrt{5} \longrightarrow c = 0$$
$$\boxed{x = -22t - 55}$$

(b)

$$y' + \frac{2}{3}x^2y = 0$$

$$W = e^{\int \frac{2}{3}x^2dx}$$
$$W = e^{2x^3/9}$$

Best choice

$$(Wy)'/W = 0$$
$$Wy = c$$
$$\boxed{y = c\,e^{-2x^3/9}}$$

(c)

$$\boxed{\begin{aligned} y_p &= -2/5 \\ y_h &= c\,e^{5x} \end{aligned}}$$

equilibrium solution

Growth-Decay Sol of $y' - 5y = 0$

Thm Sol of $y' = ky$ is $y = ce^{kx}$

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2250 Exam 1 Ver 1 S2008 [7:30]
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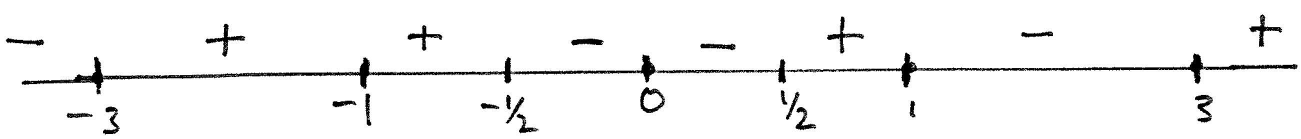
5. (Stability)

(a) [50%] Draw a phase line diagram for the differential equation

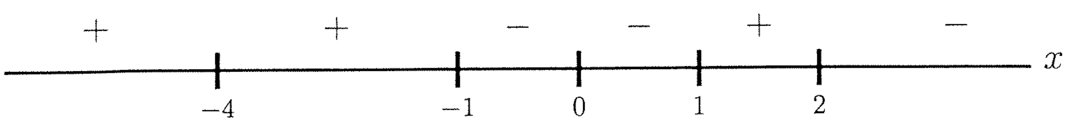
$$\frac{dx}{dt} = \ln(1+x^2)\left(1 - \sqrt[4]{|2x|}\right)^3 (1+x)(9-x^2)(x^2-1)^3.$$

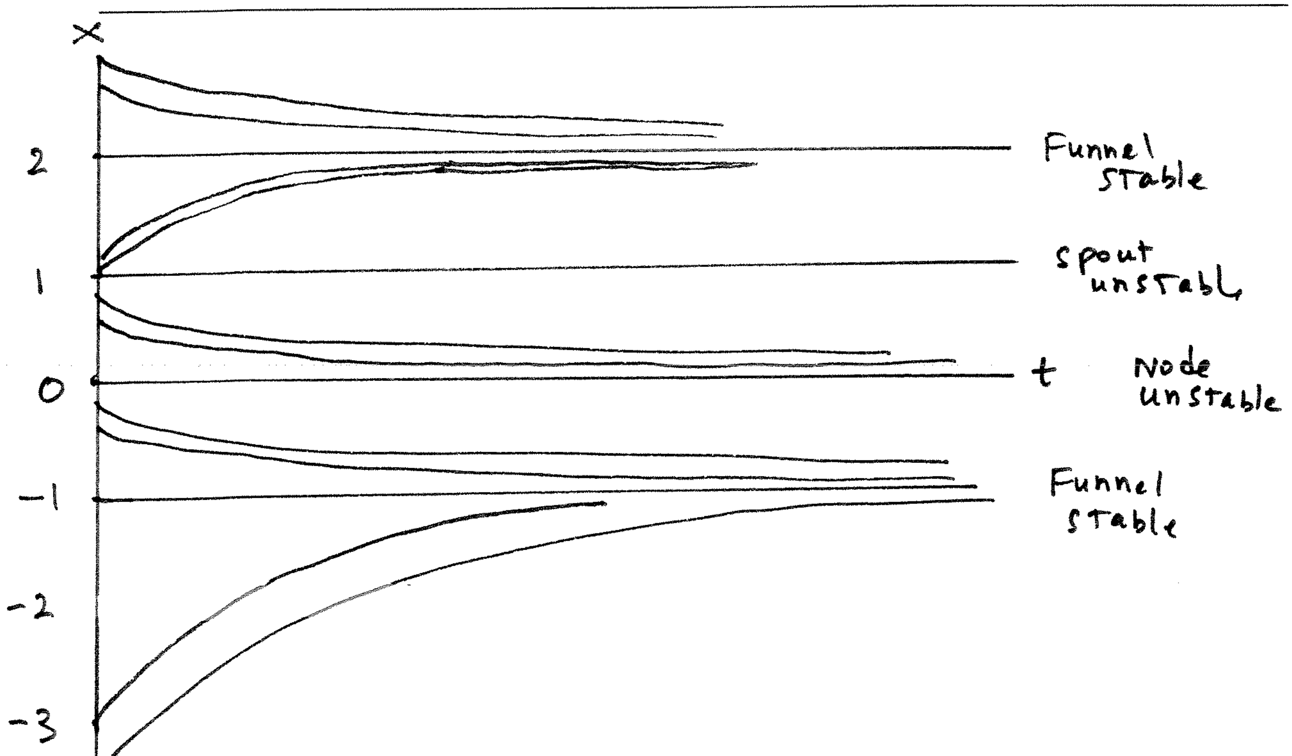
Expected in the phase line diagram are equilibrium points and signs of dx/dt.

7 equilibria



(b) [50%] Draw a phase diagram with at least 10 threaded curves, using the phase line diagram given below. Add these labels as appropriate: funnel, spout, neither spout nor funnel [a node], stable, unstable. A direction field is not expected nor required.





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