**Name** ____________________________ Class Time ______

**Math 2250 Extra Credit Problems**
**Chapter 7**
**January 2008**

**Due date**: Submit these problems on the first day of final week, under the door 113 JWB, before 9pm. Records are locked on that date and only corrected, never appended.

**Submitted work**. Please submit one stapled package per problem. Kindly label problems **Extra Credit**. Label each problem with its corresponding problem number, e.g., Xc7.1-8. You may attach this printed sheet to simplify your work.

## Problem Xc7.1-8. (Transform to a first order system)

Use the position-velocity substitution $u_1 = x(t)$, $u_2 = x'(t)$, $u_3 = y(t)$, $u_4 = y'(t)$ to transform the system below into vector-matrix form $\mathbf{u}'(t) = A\mathbf{u}(t)$. Do not attempt to solve the system.

$$x'' - 2x' + 5y = 0, \quad y'' + 2y' - 5x = 0.$$

## Problem Xc7.1-20a. (Dynamical systems)

Prove this result for system

$$\begin{array}{rcrcr} x' &=& ax &+& by, \\ y' &=& cx &+& dy. \end{array} \tag{1}$$

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**Theorem**. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and define $\textbf{trace}(A) = a + d$. Then $p_1 = -\textbf{trace}(A)$, $p_2 = \det(A)$ are the coefficients in the determinant expansion

$$\det(A - rI) = r^2 + p_1 r + p_2$$

and $x(t)$ and $y(t)$ in equation (1) are both solutions of the differential equation $u'' + p_1 u' + p_2 u = 0$.

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## Problem xC7.1-20b. (Solve dynamical systems)

(a) Apply the previous problem to solve

$$\begin{array}{rcrcr} x' &=& 2x &-& y, \\ y' &=& x &+& 2y. \end{array}$$

(b) Use first order methods to solve the system

$$\begin{array}{rcrcr} x' &=& 2x &-& y, \\ y' &=& & & 2y. \end{array}$$

## Problem Xc7.2-12. (General solution answer check)

(a) Verify that $\mathbf{x}_1(t) = e^{3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\mathbf{x}_2(t) = e^{2t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ are solutions of $\mathbf{x}' = A\mathbf{x}$, where

$$A = \begin{pmatrix} 4 & 1 \\ -2 & 1 \end{pmatrix}.$$

(b) Apply the Wronskian test $\det(\textbf{aug}(\mathbf{x}_1, \mathbf{x}2)) \neq 0$ to verify that the two solutions are independent.

(c) Display the general solution of $\mathbf{x}' = A\mathbf{x}$.

**Extra credit problems chapter 7 continue on the next page.**

**Problem Xc7.2-14. (Particular solution)**

(a) Find the constants $c_1$, $c_2$ in the general solution

$$\mathbf{x}(t) = c_1 e^{3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{5t}\begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

satisfying the initial conditions $x_1(0) = 4$, $x_2(0) = -1$.

(b) Find the matrix $A$ in the equation $\mathbf{x}' = A\mathbf{x}$. Use the formula $AP = PD$ and Fourier's model for $A$, which is given implicitly in (a) above.

**Problem Xc7.3-8. (Eigenanalysis method $2 \times 2$)**

(a) Find $\lambda_1$, $\lambda_2$, $\mathbf{v}_1$, $\mathbf{v}_2$ in Fourier's model $A\,(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2$ for

$$A = \begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix}.$$

(b) Display the general solution of $\mathbf{x}' = A\mathbf{x}$.

**Problem Xc7.3-20. (Eigenanalysis method $3 \times 3$)**

(a) Find $\lambda_1$, $\lambda_2$, $\lambda_3$, $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ in Fourier's model $A\,(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3) = c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2 + c_3\lambda_3\mathbf{v}_3$ for

$$A = \begin{pmatrix} 2 & 1 & -1 \\ -4 & -3 & -1 \\ 4 & 4 & 2 \end{pmatrix}.$$

(b) Display the general solution of $\mathbf{x}' = A\mathbf{x}$.

**Problem Xc7.3-30. (Brine Tanks)**

Consider two brine tanks satisfying the equations

$$x_1'(t) = -k_1x_1 + k_2x_2, \quad x_2' = k_1x_1 - k_2x_2.$$

Assume $r = 10$ gallons per minute, $k_1 = r/V_1$, $k_2 = r/V_2$, $x_1(0) = 30$ and $x_2(0) = 0$. Let the tanks have volumes $V_1 = 50$ and $V_2 = 25$ gallons. Solve for $x_1(t)$ and $x_2(t)$.

**Problem Xc7.3-40. (Eigenanalysis method $4 \times 4$)**

Display (a) Fourier's model and (b) the general solution of $\mathbf{x}' = A\mathbf{x}$ for the $4 \times 4$ matrix

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ -21 & -5 & -27 & -9 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & -16 & -4 \end{pmatrix}.$$

**End of extra credit problems chapter 7.**