Chapter 3

Linear Algebraic Equations

This introduction to linear algebraic equations requires only a college algebra background. Vector and matrix notation is **not used**. The subject of linear algebra, using vectors, matrices and related tools, appears later in the text; see Chapter 8.

The topics studied are *linear equations*, *general solution*, *reduced echelon system*, *basis*, *nullity*, *rank* and *nullspace*. Introduced here are the **three possibilities**, the **frame sequence**, which uses the three rules **swap**, **combination** and **multiply**, and finally the method of **elimination**, in literature called **Gauss-Jordan elimination** or **Gaussian elimination**

3.1 Linear Systems of Equations

Background from college algebra includes system of linear algebraic equations like

$$(1) \qquad \left\{ \begin{array}{rcrcl} 3x & + & 2y & = & 1, \\ x & - & y & = & 2. \end{array} \right.$$

A **solution** (x, y) of non-homogeneous system (1) is a pair of values that simultaneously satisfy both equations. This example has unique solution $x = 1$, $y = -1$.

The homogeneous system corresponding to (1) is obtained by replacing the right sides of the equations by zero:

$$(2) \qquad \left\{ \begin{array}{rcrcl} 3x & + & 2y & = & 0, \\ x & - & y & = & 0. \end{array} \right.$$

System (2) has unique solution $x = 0$, $y = 0$.

College algebra courses have emphasis on unique solutions. In this chapter we study in depth the cases for **no solution** and **infinitely many solutions**. These two cases are illustrated by the examples

No Solution

$$(3)\quad \begin{cases} x - y = 0, \\ 0 = 1. \end{cases}$$

Infinitely Many Solutions

$$(4)\quad \begin{cases} x - y = 0, \\ 0 = 0. \end{cases}$$

Equations (3) cannot have a solution because of the **signal equation** $0 = 1$, a false equation. Equations (4) have one solution (x, y) for each point on the 45° line $x - y = 0$, therefore system (4) has infinitely many solutions.

The Three Possibilities

Solutions of general linear systems with m equations in n unknowns may be classified into exactly **three possibilities**:

1. No solution.
2. Infinitely many solutions.
3. A unique solution.

General Linear Systems

Given numbers $a_{11}, \ldots, a_{mn}, b_1, \ldots, b_m$, a **nonhomogeneous system** of m linear equations in n **unknowns** $x_1, x_2, \ldots, x_n$ is the system

$$(5)\quad \begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &=& b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &=& b_2, \\ &\vdots& \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &=& b_m. \end{array}$$

Constants $a_{11}, \ldots, a_{mn}$ are called the **coefficients** of system (5). Constants $b_1, \ldots, b_m$ are collectively referenced as the **right hand side**, **right side** or **RHS**. The **homogeneous system** corresponding to system (5) is obtained by replacing the right side by zero:

$$(6)\quad \begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &=& 0, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &=& 0, \\ &\vdots& \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &=& 0. \end{array}$$

An assignment of possible values $x_1, \ldots, x_n$ which simultaneously satisfy all equations in (5) is called a **solution** of system (5). **Solving** system (5) refers to the process of finding all possible solutions of (5). The system (5) is called **consistent** if it has a solution and otherwise it is called **inconsistent**.

The Toolkit of Three Rules

Two systems (5) are said to be **equivalent** provided they have exactly the same solutions. For the purpose of solving systems, there is a toolkit of three reversible operations on equations which can be applied to obtain equivalent systems. These rules *neither create nor destroy solutions* of the original system:

Table 1. The Three Rules

Swap	Two equations can be interchanged without changing the solution set.
Multiply	An equation can be multiplied by $c \neq 0$ without changing the solution set.
Combination	A multiple of one equation can be added to a different equation without changing the solution set.

The last two rules replace an existing equation by a new one. A **swap** repeated reverses the swap operation. A **multiply** is reversed by multiplication by $1/c$, whereas the **combination** rule is reversed by subtracting the equation–multiple previously added. In short, the three operations are **reversible**.

Theorem 1 (Equivalent Systems)
A second system of linear equations, obtained from the first system of linear equations by a finite number of toolkit operations, has exactly the same solutions as the first system.

Exposition. Writing a set of equations and its equivalent system under toolkit rules demands that all equations be copied, not just the affected equation(s). Generally, each displayed system changes just one equation, the single exception being a swap of two equations. Within an equation, variables appear left-to-right in variable list order. Equations that contain no variables, typically $0 = 0$, are displayed last.

Documenting the three rules. In backboard and hand-written work, the acronyms **swap**, **mult** and **combo**, replace the longer terms *swap*, *multiply* and *combination*. They are placed next to the first changed equation. In cases where precision is required, additional information is supplied, namely the **source** and **target** equation numbers s, t and the multiplier $m \neq 0$ or c. Details:

`swap(s,t)`	Swap equations s and t.
`mult(t,m)`	Multiply target equation t by multiplier $m \neq 0$.
`combo(s,t,c)`	Multiply source equation s by multiplier c and add to target equation t.

The acronyms match usage in the computer algebra system `maple`, for package `linalg` and functions `swaprow`, `mulrow` and `addrow`.

Solving Equations with Geometry

In the plane ($n = 2$) and in 3-space ($n = 3$), equations (5) have a geometric interpretation that can provide valuable intuition about possible solutions. College algebra courses often omit the discussion of *no solutions* or *infinitely many solutions*, discussing only the case of a single unique solution. In contrast, all cases are considered here.

Plane Geometry. A **straight line** may be represented as an equation $Ax + By = C$. Solving the system

$$\begin{array}{ccccc} a_{11}x & + & a_{12}y & = & b_1 \\ a_{21}x & + & a_{22}y & = & b_2 \end{array} \tag{7}$$

is the geometrical equivalent of finding all possible (x, y)-intersections of the lines represented in system (7). The distinct geometrical possibilities appear in Figures 1–3.

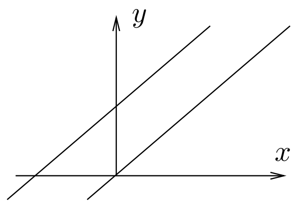

Figure 1. Parallel lines, no solution.

$$\begin{aligned} -x + y &= 1, \\ -x + y &= 0. \end{aligned}$$

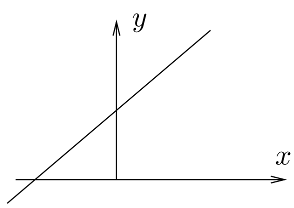

Figure 2. Identical lines, infinitely many solutions.

$$\begin{aligned} -x + y &= 1, \\ -2x + 2y &= 2. \end{aligned}$$

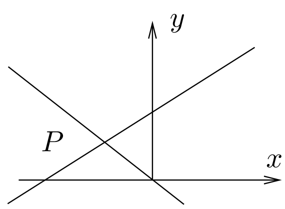

Figure 3. Non-parallel distinct lines, one solution at the unique intersection point P.

$$\begin{aligned} -x + y &= 2, \\ x + y &= 0. \end{aligned}$$

Space Geometry. A **plane** in xyz-space is given by an equation $Ax + By + Cz = D$. The vector $A\vec{\imath} + B\vec{\jmath} + C\vec{k}$ is **normal** to the plane. An equivalent equation is $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$, where (x_0, y_0, z_0) is a given point in the plane. Solving system

$$\begin{array}{ccccccc} a_{11}x & + & a_{12}y & + & a_{13}z & = & b_1 \\ a_{21}x & + & a_{22}y & + & a_{23}z & = & b_2 \\ a_{31}x & + & a_{32}y & + & a_{33}z & = & b_3 \end{array} \tag{8}$$

is the geometric equivalent of finding all possible (x, y, z)-intersections of the planes represented by system (8). Illustrated in Figures 4–11 are some interesting geometrical possibilities.

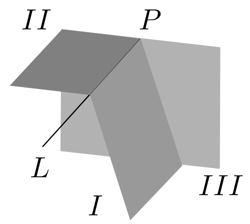

Figure 4. Knife cuts an open book. Two non-parallel planes I, II meet in a line L not parallel to plane III. There is a *unique point* P of intersection of all three planes.

$$I:\ y+z=0, \quad II:\ z=0, \quad III:\ x=0.$$

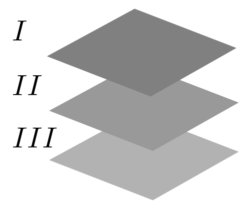

Figure 5. Triple–decker. Planes I, II, III are parallel. There is no intersection point.

$$I:\ z=2, \quad II:\ z=1, \quad III:\ z=0.$$

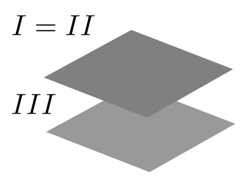

Figure 6. Double–decker. Planes I, II are equal and parallel to plane III. There is no intersection point.

$$I:\ 2z=2, \quad II:\ z=1, \quad III:\ z=0.$$

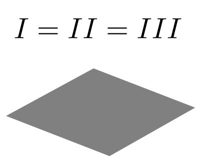

Figure 7. Single–decker. Planes I, II, III are equal. There are infinitely many intersection points.

$$I:\ z=1, \quad II:\ 2z=2, \quad III:\ 3z=3.$$

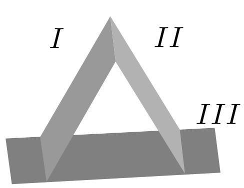

Figure 8. Pup tent. Two non-parallel planes I, II meet in a line which never meets plane III. There are no intersection points.

$$I:\ y+z=0, \quad II:\ y-z=0, \quad III:\ z=-1.$$

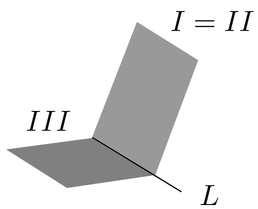

Figure 9. Open book. Equal planes I, II meet another plane III in a line L. There are infinitely many intersection points.

$$I:\ y+z=0,\ II:\ 2y+2z=0,\ III:\ z=0.$$

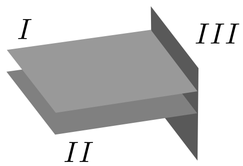

Figure 10. Book shelf. Two planes I, II are distinct and parallel. There is no intersection point.

$$I:\ z=2,\quad II:\ z=1,\quad III:\ y=0.$$

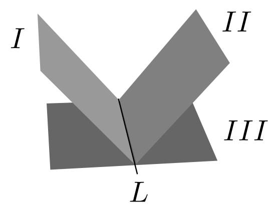

Figure 11. Saw tooth. Two non-parallel planes I, II meet in a line L which lies in a third plane III. There are infinitely many intersection points.

$$I:\ -y+z=0,\quad II:\ y+z=0,\quad III:\ z=0.$$

Examples and Methods

1 Example (Planar System) Classify the system geometrically as one of the three types displayed in Figures 1, 2, 3. Then solve for x and y.

$$\begin{array}{rcrcl} x &+& 2y &=& 1, \\ 3x &+& 6y &=& 3. \end{array} \tag{9}$$

Solution: The second equation, divided by 3, gives the first equation. In short, the two equations are proportional. The lines are geometrically **equal lines**. The two equations are equivalent to the system

$$\begin{array}{rcrcl} x &+& 2y &=& 1, \\ & & 0 &=& 0. \end{array}$$

To solve the system means to find all points (x, y) simultaneously common to both lines, which are all points (x, y) on $x + 2y = 1$.

A parametric representation of this line is possible, obtained by setting $y = t$ and then solving for $x = 1 - 2t$, $-\infty < t < \infty$. We report the solution as a parametric solution, but the first solution is also valid.

$$\begin{array}{l} x = 1 - 2t, \\ y = t. \end{array}$$

2 Example (No Solution) Classify the system geometrically as the type displayed in Figure 1. Explain why there is no solution.

$$\begin{array}{rcrcl} x &+& 2y &=& 1, \\ 3x &+& 6y &=& 6. \end{array} \tag{10}$$

Solution: The second equation, divided by 3, gives $x + 2y = 2$, a line parallel to the first line $x + 2y = 1$. The lines are geometrically **parallel lines**. The two equations are equivalent to the system

$$\begin{array}{rcrcl} x &+& 2y &=& 1, \\ x &+& 2y &=& 2. \end{array}$$

To solve the system means to find all points (x, y) simultaneously common to both lines, which are all points (x, y) on $x + 2y = 1$ and also on $x + 2y = 2$. If such a point (x, y) exists, then $1 = x + 2y = 2$ or $1 = 2$, a contradictory **signal equation**. Because $1 = 2$ is **false**, then no common point (x, y) exists and we report **no solution**.

Some readers will want to continue and write equations for x and y, a *solution* to the problem. We emphasize that this is not possible, because there is no solution at all.

The presence of a signal equation, which is a false equation used primarily to detect no solution, will appear always in the solution process for a system of equations that has no solution. Generally, this signal equation, if present, will be distilled to the single equation "$0 = 1$." For instance, $1 = 2$ can be distilled to $0 = 1$ by adding -1 across the first signal equation.

Exercises 3.1

Planar System. Solve the xy–system and interpret the solution geometrically.

1. $\begin{array}{rcrcl} x & + & y & = & 1, \\ & & y & = & 1. \end{array}$

2. $\begin{array}{rcrcr} x & + & y & = & -1, \\ x & & & = & 3. \end{array}$

3. $\begin{array}{rcrcl} x & + & y & = & 1, \\ x & + & 2y & = & 2. \end{array}$

4. $\begin{array}{rcrcl} x & + & y & = & 1, \\ x & + & 2y & = & 3. \end{array}$

5. $\begin{array}{rcrcl} x & + & y & = & 1, \\ 2x & + & 2y & = & 2. \end{array}$

6. $\begin{array}{rcrcl} 2x & + & y & = & 1, \\ 6x & + & 3y & = & 3. \end{array}$

7. $\begin{array}{rcrcr} x & - & y & = & 1, \\ -x & - & y & = & -1. \end{array}$

8. $\begin{array}{rcrcl} 2x & - & y & = & 1, \\ x & - & 0.5y & = & 0.5. \end{array}$

9. $\begin{array}{rcrcl} x & + & y & = & 1, \\ x & + & y & = & 2. \end{array}$

10. $\begin{array}{rcrcl} x & - & y & = & 1, \\ x & - & y & = & 0. \end{array}$

System in Space. For each xyz–system, interpret the solution geometrically. If there is a unique intersection point, then report the values of x, y and z.

11. $\begin{array}{rcrcrcl} x & - & y & + & z & = & 2, \\ x & & & & & = & 1, \\ & & y & & & = & 0. \end{array}$

12. $\begin{array}{rcrcrcl} x & + & y & - & 2z & = & 3, \\ x & & & & & = & 2, \\ & & & & z & = & 1. \end{array}$

13. $\begin{array}{rcrcl} x & - & y & = & 2, \\ x & - & y & = & 1, \\ x & - & y & = & 0. \end{array}$

14. $\begin{array}{rcrcl} x & + & y & = & 3, \\ x & + & y & = & 2, \\ x & + & y & = & 1. \end{array}$

15. $\begin{array}{rcrcrcl} x & + & y & + & z & = & 3, \\ x & + & y & + & z & = & 2, \\ x & + & y & + & z & = & 1. \end{array}$

16. $\begin{array}{rcrcrcl} x & + & y & + & 2z & = & 2, \\ x & + & y & + & 2z & = & 1, \\ x & + & y & + & 2z & = & 0. \end{array}$

17. $\begin{array}{rcrcrcl} x & - & y & + & z & = & 2, \\ 2x & - & 2y & + & 2z & = & 4, \\ & & y & & & = & 0. \end{array}$

18. $\begin{array}{rcrcrcl} x & + & y & - & 2z & = & 3, \\ 3x & + & 3y & - & 6z & = & 6, \\ & & & & z & = & 1. \end{array}$

19. $\begin{array}{rcl} x - y + z &=& 2, \\ 0 &=& 0, \\ 0 &=& 0. \end{array}$

20. $\begin{array}{rcl} x + y - 2z &=& 3, \\ 0 &=& 0, \\ 1 &=& 1. \end{array}$

21. $\begin{array}{rcl} x + y &=& 2, \\ x - y &=& 2, \\ y &=& -1. \end{array}$

22. $\begin{array}{rcl} x - 2z &=& 4, \\ x + 2z &=& 0, \\ z &=& 2. \end{array}$

23. $\begin{array}{rcl} y + z &=& 2, \\ 3y + 3z &=& 6, \\ y &=& 0. \end{array}$

24. $\begin{array}{rcl} x + 2z &=& 1, \\ 4x + 8z &=& 4, \\ z &=& 0. \end{array}$

3.2 Frame Sequences

We imagine ourselves watching an expert, who applies swap, multiply and combination rules to a system of equations, in order to find the solution. At each application of *swap*, *combo* or *mult*, the system of equations is re-written onto a new piece of paper.

The content of each completed paper is photographed, to produce a **frame** in a sequence of camera snapshots. The first frame is the original system and the last frame gives the solution to the system of equations.

Eliminated from the sequence are all arithmetic details, which are expected to be supplied by the reader. It is emphasized that this is not a video of the solving process, but a sequence of snapshots documenting major steps.

Table 2. A Frame Sequence.

Frame 1	Frame 2	Frame 3
Original System	Apply `mult(2,1/3)`	Apply `combo(2,1,1)`
$\begin{cases} x - y = 2, \\ 3y = -3. \end{cases}$	$\begin{cases} x - y = 2, \\ y = -1. \end{cases}$	$\begin{cases} x = 1, \\ y = -1. \end{cases}$

Lead Variables

A variable in the list x, y, z is called a **lead variable** provided it appears just once in the entire system of equations, and in addition, its appearance reading left-to-right is first, with coefficient one. The same definition applies to arbitrary variable lists, like x_1, x_2, ..., x_n.

Symbol x is a lead variable in all three frames of the sequence in Table 2. But symbol y fails to be a lead variable in frames 1 and 2. In the final frame, both x and y are lead variables.

A system without signal equations ($0 = 1$ is a typical signal equation) in which every variable is a lead variable must have a unique solution. Such a system must look like the final frame of the sequence in Table 2. More precisely, the variables appear in variable list order to the left of the equal sign, each variable appearing just once, with numbers to the right of the equal sign.

Unique Solution

To solve a system with a unique solution, we apply the toolkit operations of swap, multiply and combination (acronyms `swap, mult, combo`), one operation per frame, until the last frame displays the unique solution.

Because all variables will be lead variables in the last frame, we seek to create a new lead variable in each frame. Sometimes, this is not possible, even if it is the general objective. Exceptions are swap and multiply operations, which are often used to prepare for creation of a lead variable. Listed in Table 3 are the rules and conventions that we use to create frame sequences.

Table 3. Conventions and rules for frame sequences.

Order of Variables. Variables in equations appear in variable list order to the left of the equal sign.

Order of Equations. Equations are listed in variable list order inherited from their lead variables. Equations without lead variables appear next. Equations without variables appear last.

New Lead Variable. Among equations without a lead variable, select a new lead variable as the *first variable*, in variable list order, which appears in these equations.

$$\boxed{\begin{array}{ccccccc} & & y & + & 4z & = & 2, \\ x & + & y & & & = & 3, \\ x & + & 2y & + & 3z & = & 4. \end{array}}$$

Frame 1. Original system.

$$\boxed{\begin{array}{ccccccc} x & + & 2y & + & 3z & = & 4, \\ x & + & y & & & = & 3, \\ & & y & + & 4z & = & 2. \end{array}}$$

Frame 2.

`swap(1,3)`

$$\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 4, \\ & - & y & - & 3z & = & -1, \\ & & y & + & 4z & = & 2. \end{array}$$

Frame 3.
`combo(1,2,-1)`

$$\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 4, \\ & - & y & - & 3z & = & -1, \\ & & & & z & = & 1. \end{array}$$

Frame 4.
`combo(2,3,1)`

$$\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 4, \\ & & y & + & 3z & = & 1, \\ & & & & z & = & 1. \end{array}$$

Frame 5.
`mult(2,-1)`

$$\begin{array}{rcrcrcr} x & & & - & 3z & = & 2, \\ & & y & + & 3z & = & 1, \\ & & & & z & = & 1. \end{array}$$

Frame 6.
`combo(2,1,-2)`

$$\begin{array}{rcrcrcr} x & & & - & 3z & = & 2, \\ & & y & & & = & -2, \\ & & & & z & = & 1. \end{array}$$

Frame 7.
`combo(3,2,-3)`

$$\begin{array}{rcrcrcr} x & & & & & = & 5, \\ & & y & & & = & -2, \\ & & & & z & = & 1. \end{array}$$

Frame 8. `combo(3,1,3)`
Last Frame.
Unique solution.

No Solution

A special case occurs in a frame sequence, when a nonzero equation occurs having no variables. Called a **signal equation**, its occurrence signals **no solution**, because the equation is false. Normally, we halt the frame sequence at the point of first discovery, and then declare no solution. An illustration:

$$\begin{array}{rcrcrcr} & & y & + & 3z & = & 2, \\ x & + & y & & & = & 3, \\ x & + & 2y & + & 3z & = & 4. \end{array}$$

Frame 1. Original system.

$$\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 4, \\ x & + & y & & & = & 3, \\ & & y & + & 3z & = & 2. \end{array}$$

Frame 2.
`swap(1,3)`

$$\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 4, \\ & - & y & - & 3z & = & -1, \\ & & y & + & 3z & = & 2. \end{array}$$

Frame 3.
`combo(1,2,-1)`

$$\boxed{\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 4, \\ & - & y & - & 3z & = & -1, \\ & & & & 0 & = & 1. \end{array}}$$

Frame 4.
Signal Equation $0 = 1$.
`combo(2,3,1)`

The signal equation $\boxed{0 = 1}$ is a false equation, therefore the last frame has no solution. Because the toolkit neither creates nor destroys solutions, then the first frame (the original system) has **no solution**.

Readers who want to go on and write an answer for the system must be warned that no such possibility exists. Values cannot be assigned to any variables in the case of no solution. This can be perplexing, especially in a final frame like

$$\boxed{\begin{array}{lcr} x & = & 4, \\ z & = & -1, \\ 0 & = & 1. \end{array}}$$

While it is true that x and z were assigned values, the final signal equation $0 = 1$ is false, meaning any answer is impossible. There is no possibility to write equations for all variables. There is **no solution**.

Infinitely Many Solutions

A system of equations having infinitely many solutions is solved from a frame sequence construction that parallels the unique solution case. The same quest for lead variables is made, hoping in the final frame to have just the variable list on the left and numbers on the right.

The stopping criterion which identifies the final frame, in either the case of a unique solution or infinitely many solutions, is exactly the same:

> The last frame is attained when every nonzero equation has a lead variable. Remaining equations must be of the form $0 = 0$.

Any variables that are not lead variables, in the final frame, are called **free variables**, because their values are completely undetermined.

$$\boxed{\begin{array}{rcrcrcr} & & y & + & z & = & 1, \\ x & + & y & & & = & 3, \\ x & + & 2y & + & 3z & = & 4. \end{array}}$$

Frame 1. Original system.

$$\boxed{\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 4, \\ x & + & y & & & = & 3, \\ & & y & + & 3z & = & 1. \end{array}}$$

Frame 2.

`swap(1,3)`

$$\boxed{\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 4,\\ & - & y & - & 3z & = & -1,\\ & & y & + & 3z & = & 1.\end{array}}$$

Frame 3.
`combo(1,2,-1)`

$$\boxed{\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 4,\\ & - & y & - & 3z & = & -1,\\ & & & & 0 & = & 0.\end{array}}$$

Frame 4.

`combo(2,3,1)`

$$\boxed{\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 4,\\ & & y & + & 3z & = & 1,\\ & & & & 0 & = & 0.\end{array}}$$

Frame 5.
`mult(2,-1)`

$$\boxed{\begin{array}{rcrcrcr} x & & & - & 3z & = & 2,\\ & & y & + & 3z & = & 1,\\ & & & & 0 & = & 0.\end{array}}$$

Frame 6. `combo(2,1,-2)`
Last Frame.
Lead variables x, y.

Last Frame to General Solution

Once the *last frame* of the frame sequence is obtained, then the general solution can be written by a fixed and easy-to-learn algorithm. This process is used only in case of infinitely many solutions.

(1) Assign invented symbols t_1, t_2, ... to the free variables.[1]

(2) Isolate each lead variable.

(3) Back-substitute the free variable invented symbols.

From the last frame of the frame sequence,

$$\boxed{\begin{array}{rcrcrcr} x & & & - & 3z & = & 2,\\ & & y & + & 3z & = & 1,\\ & & & & 0 & = & 0,\end{array}}$$

Last Frame.
Lead variables x, y.

the general solution is written as follows.

$$\boxed{z = t_1}$$

The free variable z is assigned symbol t_1.

$$\boxed{\begin{array}{l} x = 2 + 3z,\\ y = 1 - 3z\end{array}}$$

The lead variables are x, y. Isolate them left.

$$\boxed{\begin{array}{l} x = 2 + 3t_1,\\ y = 1 - 3t_1,\\ z = t_1.\end{array}}$$

Back-substitute. Solution found.

[1] Computer algebra system `maple` uses these invented symbols, hence our convention here is to use t_1, t_2, t_3, ... as the list of invented symbols.

The solution found in the last step is called a **standard general solution**. The meaning is that all solutions of the system of equations can be found by specializing the invented symbols t_1, t_2, ... to particular numbers. Also implied is that the general solution expression satisfies the system of equations for all possible values of the symbols t_1, t_2, ...

3.3 General Solution Theory

The **general solution** of a system of linear algebraic equations is an expression which represents all possible solutions of the system.

The example above for infinitely many solutions contained an unmotivated algorithm which expressed the general solution in terms of invented symbols t_1, t_2, ..., which in mathematical literature are called **parameters**. We outline here some topics from calculus which form the assumed background for this subject.

Equations for Points, Lines and Planes

Background from analytic geometry appears in Table 4. In this table, t_1 and t_2 are **parameters**, which means they are allowed to take on any value between $-\infty$ and $+\infty$. The algebraic equations describing the geometric objects are called **parametric equations**.

Table 4. Parametric equations with geometrical significance.

$x = d_1$, $y = d_2$, $z = d_3$.	**Point**. The equations have no parameters and describe a single point.
$x = d_1 + a_1 t_1$, $y = d_2 + a_2 t_1$, $z = d_3 + a_3 t_1$.	**Line**. The equations with parameter t_1 describe a straight line through (d_1, d_2, d_3) with tangent vector $a_1\vec{\imath} + a_2\vec{\jmath} + a_3\vec{k}$.
$x = d_1 + a_1 t_1 + b_1 t_2$, $y = d_2 + a_2 t_1 + b_2 t_2$, $z = d_3 + a_3 t_1 + b_3 t_2$.	**Plane**. The equations with parameters t_1, t_2 describe a plane containing (d_1, d_2, d_3). The cross product $(a_1\vec{\imath} + a_2\vec{\jmath} + a_3\vec{k}) \times (b_1\vec{\imath} + b_2\vec{\jmath} + b_3\vec{k})$ is normal to the plane.

To illustrate, the parametric equations $x = 2 - 6t_1$, $y = -1 - t_1$, $z = 8t_1$ describe the unique line of intersection of the three planes

$$\begin{array}{rcrcrcl} x &+& 2y &+& z &=& 0, \\ 2x &-& 4y &+& z &=& 8, \\ 3x &-& 2y &+& 2z &=& 8. \end{array} \tag{11}$$

Details appear in Example 3.

General Solutions

Definition 1 (Parametric Equations)
Equations of the form

$$
\begin{aligned}
x_1 &= d_1 + c_{11}t_1 + \cdots + c_{1k}t_k, \\
x_2 &= d_2 + c_{21}t_1 + \cdots + c_{2k}t_k, \\
&\vdots \\
x_n &= d_n + c_{n1}t_1 + \cdots + c_{nk}t_k
\end{aligned} \tag{12}
$$

are called **parametric equations**.

The numbers $d_1, \ldots, d_n, c_{11}, \ldots, c_{nk}$ are *known constants* and the variable names $t_1, \ldots, t_k$ are **parameters**. The symbols $t_1, \ldots, t_k$ are allowed to take on any value from $-\infty$ to ∞.

Definition 2 (General Solution)
A **general solution** of a linear algebraic system of equations (5) is a set of parametric equations (12) plus two additional requirements:

(13) Equations (12) satisfy (5) for all real values of $t_1, \ldots, t_k$.

(14) Any solution of (5) can be obtained from (12) by specializing values of the parameters $t_1, t_2, \ldots t_k$.

A general solution is sometimes called a **parametric solution**. Requirements (13), (14) mean that the solution works and we didn't skip any solutions.

Definition 3 (Standard General Solution)
Parametric equations (12) are called **standard** if they satisfy for distinct subscripts $j_1, i_2, \ldots, j_k$ the equations

$$x_{j_1} = t_1, \quad x_{j_2} = t_2, \quad \ldots, \quad x_{j_k} = t_k. \tag{15}$$

The relations mean that the full set of parameter symbols $t_1, t_2, \ldots, t_k$ were assigned to k distinct variable names selected from $x_1, \ldots, x_n$.

A **standard general solution** of system (5) is a special set of parametric equations (12) satisfying (13), (14) and additionally (15). Frame sequences always produce a standard general solution.

Theorem 2 (Standard General Solution)
A standard general solution has the fewest possible parameters and it represents each solution of the linear system by a unique set of parameter values.

The theorem supplies the theoretical basis for the method of frame sequences, which formally appears as an algorithm on page 179. The proof of Theorem 2 is delayed until page 197. It is unlikely that this proof will be a subject of a class lecture, due to its length; it is recommended reading for the mathematically inclined, after understanding the examples.

Reduced Echelon System

The last frame of a sequence which has a unique solution or infinitely many solutions is called a **reduced echelon system**. The definition is repeated, for clarity and precision. We consider a system of linear algebraic equations and explain how to classify it as the last of a frame sequence.

> A variable is called a **lead variable** provided it appears first and with coefficient one in exactly one equation.
>
> A linear system in which each nonzero equation has a **lead variable** is called a **reduced echelon system**. Such equations by convention are listed by variable list order inherited from their lead variables. Any zero equations are listed last.
>
> A variable in a reduced echelon system not a lead variable is called a **free variable**. All variables in an equation are required to appear in variable list order, therefore free variables follow lead variables.

Rank and Nullity

A reduced echelon system splits the variable names $x_1, \ldots, x_n$ into two sets of variables called **lead variables** and **free variables**. The subscripts 1, 2, ..., n are split into two groups $i_1 < i_2 < \cdots < i_m$ (lead variable subscripts) and $j_1 < j_2 < \cdots < j_k$ (free variable subscripts). The variable list order is retained in each. Every subscript is listed exactly once, which means $n = m + k$. In summary:

(16)

Lead variables	**Free variables**
x_{i_1}	x_{j_1}
x_{i_2}	x_{j_2}
$\vdots$	$\vdots$
x_{i_m}	x_{j_k}
$n = m + k$	

Definition 4 (Rank and Nullity)
The number of lead variables in a reduced echelon system is called the **rank** of the system. More generally, the **rank** of a system is the rank of a final frame in any frame sequence starting with that system.

The number of free variables in a reduced echelon system is called the **nullity** of the system. More generally, the **nullity** of a system is the nullity of a final frame in any frame sequence starting with that system.

Theorem 3 (Rank and Nullity)
The following equation holds:

$$\mathbf{rank} + \mathbf{nullity} = \text{number of variables.}$$

Detecting a Reduced Echelon System

There is a way to inspect a given system, to tell if it can be easily changed into a reduced echelon system. We assume that within each equation, variables appear in variable list order.

> A linear system (5) is recognized as a reduced echelon system when the first variable listed in each equation has coefficient one and that symbol appears nowhere else in the system of equations.

Such a system can be re-written, by swapping equations, so that the resulting system is a reduced echelon system.

Computers and Reduced Echelon Form

Computer algebra systems and computer numerical laboratories compute from a given linear system (5) a new equivalent system of identical size, which is called the **reduced row-echelon form**, abbreviated **rref**.

For a consistent linear system, the new system is the reduced echelon system. A frame sequence starting with the original system has this system as its last frame.

For an inconsistent system, we allow the appearance of a signal equation $0 = 1$, a false equation used primarily to detect inconsistency. There is only one signal equation allowed, and it immediately precedes any $0 = 0$ equations.

Every computer-produced `rref` for a consistent system is a **reduced echelon system**. For inconsistent systems, the computer-produced `rref` gives a final frame with a signal equation, causing us to halt the sequence and report no solution.

To use computer assist requires matrix entry of the data, a topic which is delayed until a later chapter. Popular commercial programs used to perform the computer assist are `maple`, `mathematica` and `matlab`.

Writing a Standard General Solution

An additional illustration will be given to explain the idea. Assume **variable list order** x, w, u, v, y, z for the **reduced echelon system**

$$\begin{array}{lrcl} (17) \qquad & \boxed{x} + 4w + u + v & = & 1, \\ & \boxed{y} - u + v & = & 2, \\ & \boxed{z} - w + 2u - v & = & 0. \end{array}$$

The **lead variables** in (17) are the boxed symbols x, y, z. The **free variables** are w, u, v. Assign invented symbols t_1, t_2, t_3 to the free variables and back-substitute in (17) to obtain a **standard general solution**

$$\left\{\begin{array}{lcl} x & = & 1 - 4t_1 - t_2 - t_3, \\ y & = & 2 + t_2 - t_3 \\ z & = & t_1 - 2t_2 + t_3, \\ w & = & t_1, \\ u & = & t_2, \\ v & = & t_3. \end{array}\right. \quad \text{or} \quad \left\{\begin{array}{lcl} x & = & 1 - 4t_1 - t_2 - t_3, \\ w & = & t_1, \\ u & = & t_2, \\ v & = & t_3, \\ y & = & 2 + t_2 - t_3, \\ z & = & t_1 - 2t_2 + t_3. \end{array}\right.$$

It is demanded by convention that general solutions be displayed in variable list order. This is why the above display bothers to re-write the equations in the new order on the right.

Elimination

This algorithm applies, at each algebraic step, one of the three toolkit rules defined in Table 1: **swap**, **multiply** and **combination**.

The objective of each algebraic step is to **increase the number of lead variables**. The process stops when no more lead variables can be found, in which case the last system of equations is a **reduced echelon system**. A detailed explanation of the process has been given above in the discussion of frame sequences.

Reversibility of the algebraic steps means that no solutions are created nor destroyed throughout the algebraic steps: the original system and all systems in the intermediate steps have *exactly the same solutions.*

The final reduced echelon system has an easily–found standard general solution which is reported as the **general solution**.

Theorem 4 (Elimination)
Every linear system (5) has either no solution or else it has exactly the same solutions as an equivalent reduced echelon system, obtained by repeated application of **swap**, **multiply** and **combination** (page 165).

An Elimination Algorithm

An equation is said to be **processed** if it has a lead variable. Otherwise, the equation is said to be **unprocessed**.

The acronym `rref` abbreviates the phrase *reduced row echelon form*. This abbreviation appears in matrix literature, so we use it instead of creating an acronym `ref` (for *reduced echelon form*).

1. If an equation "$0 = 0$" appears, then move it to the end. If a signal equation "$0 = c$" appears ($c \neq 0$ required), then the system is inconsistent and the algorithm halts. We then report **no solution** and stop.
2. Identify the **first symbol** x_r, in variable list order $x_1, \ldots, x_n$, which appears in some unprocessed equation. Apply the **multiply** rule to insure x_r has leading coefficient one. Apply the **combination** rule to eliminate variable x_r from all other equations. Then x_r is a **lead variable**: the number of lead variables has been increased by one.
3. Apply the **swap** rule repeatedly to move this equation past all processed equations, but before the unprocessed equations. Mark the equation as **processed**, e.g., replace x_r by boxed symbol $\boxed{x_r}$.
4. Repeat steps 1–3, until all equations have been processed once. Then lead variables $x_{i_1}, \ldots, x_{i_m}$ have been defined and the last system is a reduced echelon system.

Uniqueness, Lead Variables and RREF

Gaussian elimination performed on a given system by two different persons will result in the same reduced echelon system. The answer is unique, because attention has been paid to the natural order of the variable list $x_1, \ldots, x_n$. Uniqueness results from *critical step* **2**, also called the **rref step**:

> Always select a lead variable as the next possible variable name in the original list order $x_1, \ldots, x_n$, taken from all possible unprocessed equations.

This step insures that the final system is in **reduced echelon form**.

The wording **next possible** must be used, because once a variable name is used for a lead variable it may not be used again. The next variable following the last–used lead variable, from the list $x_1, \ldots, x_n$, might not appear in any unprocessed equation, in which case it is a **free variable**. The next variable name in the original list order is then tried as a lead variable.

Avoiding Fractions

Integer arithmetic should be used, when possible, to speed up hand computation in Gaussian elimination. To avoid fractions, the **rref** step **2** may be modified to read *with leading coefficient nonzero.* The final division to obtain leading coefficient one is then delayed until last.

Examples and Methods

3 **Example (Line of Intersection)** Show that the parametric equations $x = 2-6t$, $y=-1-t$, $z=8t$ represent a line through $(2,-1,0)$ with tangent $-6\vec{\imath}-\vec{\jmath}$ which is the line of intersection of the three planes

$$\begin{array}{rcrcrcl} x &+& 2y &+& z &=& 0, \\ 2x &-& 4y &+& z &=& 8, \\ 3x &-& 2y &+& 2z &=& 8. \end{array} \tag{18}$$

Solution: Using $t=0$ in the parametric solution shows that $(2,-1,0)$ is on the line. The tangent to the parametric curve is $x'(t)\vec{\imath}+y'(t)\vec{\jmath}+z'(t)\vec{k}$, which computes to $-6\vec{\imath}-\vec{\jmath}$. The details for showing the parametric solution satisfies the three equations simultaneously:

$\text{LHS} = x+2y+z$	First equation left side.
$= (2-6t)+2(-1-t)+8t$	Substitute parametric solution.
$= 0$	Matches the RHS in (18).
$\text{LHS} = 2x-4y+z$	Second equation left side.
$= 2(2-6t)-4(-1-t)+8t$	Substitute.
$= 8$	Matches (18).
$\text{LHS} = 3x-2y+2z$	Third equation left side.
$= 3(2-6t)-2(-1-t)+16t$	Substitute.
$= 8$	Matches (18).

4 **Example (Geometry of Solutions)** Solve the system and interpret the solution geometrically.

$$\begin{array}{rcrcl} y &+& z &=& 1, \\ x &+& 2z &=& 3. \end{array}$$

Solution: We begin by displaying the general solution, which is a **line**:

$$\begin{array}{rcll} x &=& 3-2t_1, & \\ y &=& 1-t_1, & \\ z &=& t_1, & -\infty < t_1 < \infty. \end{array}$$

In standard xyz-coordinates, this line passes through $(3,1,0)$ with tangent direction $-2\vec{\imath}-\vec{\jmath}+\vec{k}$.

Details. To justify this solution, we observe that the first frame equals the last frame, which is a reduced echelon system. The standard general solution will be obtained from the last frame.

The variable list has six possible orderings, but the order of appearance y, z, x will be used in this example.

$$\boxed{\begin{array}{rcrcl} y & + & z & = & 1, \\ x & + & 2z & = & 3. \end{array}}$$

Frame 1 equals the last frame, a reduced echelon system The lead variables are y, x and the free variable is z.

$$\boxed{\begin{array}{rclcr} y & = & 1 & - & z, \\ x & = & 3 & - & 2z, \\ z & = & t_1. & & \end{array}}$$

Assign to z invented symbol t_1. Solve for lead variables y and x in terms of the free variable z.

$$\boxed{\begin{array}{rclcr} y & = & 1 & - & t_1, \\ x & = & 3 & - & 2t_1, \\ z & = & t_1. & & \end{array}}$$

Back-substitute for free variable z. This is the standard general solution. It is geometrically a line, by Table 4.

5 Example (Symbolic Answer Check) Perform an answer check on

$$\begin{array}{rcrcl} y & + & z & = & 1, \\ x & + & 2z & = & 3, \end{array}$$

for the general solution

$$\begin{array}{rcl} x & = & 3 - 2t_1, \\ y & = & 1 - t_1, \\ z & = & t_1, \qquad -\infty < t_1 < \infty. \end{array}$$

Solution: The displayed answer can be checked manually by substituting the symbolic general solution into the equations $y + z = 1$, $x + 2z = 3$, as follows:

$$\begin{array}{rcl} y + z & = & (1 - t_1) + (t_1) \\ & = & 1, \\ x + 2z & = & (3 - 2t_1) + 2(t_1) \\ & = & 3. \end{array}$$

Therefore, the two equations are satisfied for all values of the symbol t_1.

Errors and Skipped Solutions. An algebraic error could lead to a claimed solution $x = 3$, $y = 1$, $z = 0$, which also passes the answer check. While it is *true* that $x = 3$, $y = 1$, $z = 0$ is a solution, it is not the general solution. Infinitely many solutions were skipped in the answer check.

General Solution and Free Variables. The number of lead variables is called the **rank**. The number of free variables equals the number of variables minus the rank. Computer algebra systems can compute the rank independently, as a double-check against hand computation. This check is useful for discovering skipped solution errors. The value of the rank is unaffected by the ordering of variables.

6 Example (Elimination) Solve the system.

$$\begin{array}{ccccccccc} w & + & 2x & - & y & + & z & = & 1, \\ w & + & 3x & - & y & + & 2z & = & 0, \\ & & x & & & + & z & = & -1. \end{array}$$

Solution: The answer using the natural variable list order w, x, y, z is the standard general solution

$$\begin{array}{rcll} w & = & 3+t_1+t_2, & \\ x & = & -1-t_2, & \\ y & = & t_1, & \\ z & = & t_2, & \quad -\infty < t_1, t_2 < \infty. \end{array}$$

Details. Elimination will be applied to obtain a frame sequence whose last frame justifies the reported solution. The details amount to applying the three rules **swap**, **multiply** and **combination** for equivalent equations on page 165 to obtain a last frame which is a reduced echelon system. The standard general solution for the last frame matches the one reported above.

Let's mark selected lead variables with an asterisk (w is marked w^*) and then mark processed equations with a box enclosing the lead variable (w is marked $\boxed{w}$).

$$\boxed{\begin{array}{ccccccccc} w & + & 2x & - & y & + & z & = & 1 \\ w & + & 3x & - & y & + & 2z & = & 0 \\ & & x & & & + & z & = & -1 \end{array}} \quad \boxed{\mathbf{1}}$$

$$\boxed{\begin{array}{ccccccccc} w^* & + & 2x & - & y & + & z & = & 1 \\ 0 & + & x & + & 0 & + & z & = & -1 \\ & & x & & & + & z & = & -1 \end{array}} \quad \boxed{\mathbf{2}}$$

$$\boxed{\begin{array}{ccccccccc} \boxed{w} & + & 2x & - & y & + & z & = & 1 \\ & & x^* & & & + & z & = & -1 \\ & & & & & & 0 & = & 0 \end{array}} \quad \boxed{\mathbf{3}}$$

$$\boxed{\begin{array}{ccccccccc} \boxed{w} & + & 0 & - & y & - & z & = & 3 \\ & & \boxed{x} & & & + & z & = & -1 \\ & & & & & & 0 & = & 0 \end{array}} \quad \boxed{\mathbf{4}}$$

$\boxed{\mathbf{1}}$ Original system. Identify the variable order as w, x, y, z.

$\boxed{\mathbf{2}}$ Choose w as a lead variable. Eliminate w from equation 2 by using `combo(1,2,-1)`.

$\boxed{\mathbf{3}}$ The w-equation is processed. Let x be the next lead variable. Eliminate x from equation 3 using `combo(2,3,-1)`.

$\boxed{\mathbf{4}}$ Eliminate x from equation 1 using `combo(2,1,-2)`. Mark the x-equation as processed. **Reduced echelon system** found.

The four frames make the **frame sequence** which takes the original system into a reduced echelon system. Basic exposition rules apply:

1. Variables in an equation appear in variable list order.
2. Equations inherit variable list order from the lead variables.

The last frame of the sequence is used to write out the general solution, as follows.

$$\begin{array}{rcrcrcr} \boxed{w} &=& 3 &+& y &+& z \\ \boxed{x} &=& -1 &-& z && \\ y &=& t_1 &&&& \\ z &=& t_2 &&&& \end{array}$$

Solve for the lead variables $\boxed{w}$, $\boxed{x}$. Assign invented symbols t_1, t_2 to the free variables y, z.

$$\begin{array}{rcrcrcr} w &=& 3 &+& t_1 &+& t_2 \\ x &=& -1 &-& t_2 && \\ y &=& t_1 &&&& \\ z &=& t_2 &&&& \end{array}$$

Back-substitute free variables into the lead variable equations to get a standard general solution.

Answer check. The check will be performed according to the outline on page 195. The justification for this forward reference is to illustrate how to check answers without using the invented symbols t_1, t_2, ... in the details.

Step 1. The **nonhomogeneous trial solution** $w = 3$, $x = -1$, $y = z = 0$ is obtained by setting $t_1 = t_2 = 0$. It is required to satisfy the nonhomogeneous system

$$\begin{array}{rcrcrcrcr} w &+& 2x &-& y &+& z &=& 1, \\ w &+& 3x &-& y &+& 2z &=& 0, \\ && x &&&+& z &=& -1. \end{array}$$

Step 2. The partial derivatives ∂_{t_1}, ∂_{t_2} are applied to the parametric solution to obtain two homogeneous trial solutions $w = 1$, $x = 0$, $y = 1$, $z = 0$ and $w = 1$, $x = -1$, $y = 0$, $z = 1$, which are required to satisfy the homogeneous system

$$\begin{array}{rcrcrcrcr} w &+& 2x &-& y &+& z &=& 0, \\ w &+& 3x &-& y &+& 2z &=& 0, \\ && x &&&+& z &=& 0. \end{array}$$

Each trial solution from **Step 1** and **Step 2** is checked by direct substitution.

7 Example (No solution) Verify by applying elimination that the system has no solution.

$$\begin{array}{rcrcrcrcr} w &+& 2x &-& y &+& z &=& 0, \\ w &+& 3x &-& y &+& 2z &=& 0, \\ && x &&&+& z &=& 1. \end{array}$$

Solution: The elimination algorithm (page 180) will be applied, using the three rules **swap**, **multiply** and **combination** for equivalent equations (page 165).

$$\begin{array}{cccccccccc} w^* & + & 2x & - & y & + & z & = & 0 \\ w & + & 3x & - & y & + & 2z & = & 0 \\ & & x & & & + & z & = & 1 \end{array} \quad \boxed{1}$$

$$\begin{array}{cccccccccc} \boxed{w} & + & 2x & - & y & + & z & = & 0 \\ 0 & + & x & + & 0 & + & z & = & 0 \\ & & x & & & + & z & = & 1 \end{array} \quad \boxed{2}$$

$$\begin{array}{cccccccccc} \boxed{w} & + & 2x & - & y & + & z & = & 0 \\ & & x^* & & & + & z & = & 0 \\ & & & & & & 0 & = & 1 \end{array} \quad \boxed{3}$$

$\boxed{1}$ Original system. Select variable order w, x, y, z. Identify lead variable w, then mark it w^*.

$\boxed{2}$ Eliminate w from other equations using `combo(1,2,-1)`. Mark the w-equation processed with $\boxed{w}$.

$\boxed{3}$ Identify lead variable x, then mark it x^*. Eliminate x from the third equation using `combo(2,3,-1)`.

The appearance of the signal equation "$0 = 1$" means **no solution**. The logic: if the original system has a solution, then so does the present equivalent system, hence $0 = 1$, a contradiction. Elimination halts, because of the **inconsistent system** containing the false equation "$0 = 1$."

8 Example (Reduced Echelon form) Find an equivalent system in reduced echelon form.

$$\begin{array}{ccccccccc} x_1 & + & 2x_2 & - & x_3 & + & x_4 & = & 1, \\ x_1 & + & 3x_2 & - & x_3 & + & 2x_4 & = & 0, \\ & & x_2 & & & + & x_4 & = & -1. \end{array}$$

Solution: The answer using the natural variable list order x_1, x_2, x_2, x_4 is the non-homogeneous system in **reduced echelon form** (briefly, **rref** form)

$$\begin{array}{ccccccccc} x_1 & & & - & x_3 & - & x_4 & = & 3 \\ & & x_2 & & & + & x_4 & = & -1 \\ & & & & & & 0 & = & 0 \end{array}$$

The **lead variables** are x_1, x_2 and the **free variables** are x_3, x_4. The standard general solution of this system is

$$\begin{aligned} x_1 &= 3 + t_1 + t_2, \\ x_2 &= -1 - t_2, \\ x_3 &= t_1, \\ x_4 &= t_2, \qquad\qquad -\infty < t_1, t_2 < \infty. \end{aligned}$$

The details are the same as Example 6, with $w = x_1$, $x = x_2$, $y = x_3$, $z = x_4$. The frame sequence has three frames and the last frame is used to display the general solution.

Answer check in `maple`. The output below duplicates the reduced echelon system reported above and the general solution.

```
with(LinearAlgebra):
 eq1:=x[1]+2*x[2]-x[3]+x[4]=1:eq2:=x[1]+3*x[2]-x[3]+2*x[4]=0:
 eq3:=x[2]+x[4]=-1:eqs:=[eq1,eq2,eq3]:var:=[x[1],x[2],x[3],x[4]]:
 A:=GenerateMatrix(eqs,var,augmented);
 F:=ReducedRowEchelonForm(A);
 GenerateEquations(F,var);
 F,LinearSolve(F,free=t); # general solution answer check
 A,LinearSolve(A,free=t); # general solution answer check
```

Exercises 3.3

Classification. Classify the parametric equations as a point, line or plane, then compute as appropriate the tangent to the line or the normal to the plane.

1. $x = 0$, $y = 1$, $z = -2$

2. $x = 1$, $y = -1$, $z = 2$

3. $x = t_1$, $y = 1 + t_1$, $z = 0$

4. $x = 0$, $y = 0$, $z = 1 + t_1$

5. $x = 1 + t_1$, $y = 0$, $z = t_2$

6. $x = t_2 + t_1$, $y = t_2$, $z = t_1$

7. $x = 1$, $y = 1 + t_1$, $z = 1 + t_2$

8. $x = t_2 + t_1$, $y = t_1 - t_2$, $z = 0$

9. $x = t_2$, $y = 1 + t_1$, $z = t_1 + t_2$

10. $x = 3t_2 + t_1$, $y = t_1 - t_2$, $z = 2t_1$

Reduced Echelon System. Solve the xyz–system and interpret the solution geometrically.

11. $\begin{aligned} y + z &= 1, \\ x + 2z &= 2. \end{aligned}$

12. $\begin{aligned} x + z &= 1, \\ y + 2z &= 4. \end{aligned}$

13. $\begin{aligned} y + z &= 1, \\ x + 3z &= 2. \end{aligned}$

14. $\begin{aligned} x + z &= 1, \\ y + z &= 5. \end{aligned}$

15. $\begin{aligned} x + z &= 1, \\ 2x + 2z &= 2. \end{aligned}$

16. $\begin{aligned} x + y &= 1, \\ 3x + 3y &= 3. \end{aligned}$

17. $x + y + z = 1.$

18. $x + 2y + 4z = 0.$

19. $\begin{aligned} x + y &= 2, \\ z &= 1. \end{aligned}$

20. $\begin{aligned} x + 4z &= 0, \\ y &= 1. \end{aligned}$

Homogeneous System. Solve the xyz–system using the Gaussian algorithm and variable order x, y, z.

21. $\begin{aligned} y + z &= 0, \\ 2x + 2z &= 0. \end{aligned}$

22. $\begin{aligned} x + z &= 0, \\ 2y + 2z &= 0. \end{aligned}$

23. $\begin{aligned} x + z &= 0, \\ 2z &= 0. \end{aligned}$

24. $\begin{aligned} y + z &= 0, \\ y + 3z &= 0. \end{aligned}$

25. $x + 2y + 3z = 0.$

26. $x + 2y = 0.$

27. $\begin{aligned} y + z &= 0, \\ 2x + 2z &= 0, \\ x + z &= 0. \end{aligned}$

28. $\begin{aligned} 2x + y + z &= 0, \\ x + 2z &= 0, \\ x + y - z &= 0. \end{aligned}$

29. $$\begin{array}{rcrcrcr} x &+& y &+& z &=& 0, \\ 2x && &+& 2z &=& 0, \\ x && &+& z &=& 0. \end{array}$$

30. $$\begin{array}{rcrcrcr} x &+& y &+& z &=& 0, \\ 2x && &+& 2z &=& 0, \\ 3x &+& y &+& 3z &=& 0. \end{array}$$

Nonhomogeneous System. Solve the system using variable order x, y, z.

31. $$\begin{array}{rcrcr} y && &=& 1, \\ && 2z &=& 2. \end{array}$$

32. $$\begin{array}{rcrcr} x && &=& 1, \\ && 2z &=& 2. \end{array}$$

33. $$\begin{array}{rcrcrcr} && y &+& z &=& 1, \\ 2x && &+& 2z &=& 2, \\ x && &+& z &=& 1. \end{array}$$

34. $$\begin{array}{rcrcrcr} 2x &+& y &+& z &=& 1, \\ x && &+& 2z &=& 2, \\ x &+& y &-& z &=& -1. \end{array}$$

35. $$\begin{array}{rcrcrcr} x &+& y &+& z &=& 1, \\ 2x && &+& 2z &=& 2, \\ x && &+& z &=& 1. \end{array}$$

36. $$\begin{array}{rcrcrcr} x &+& y &+& z &=& 1, \\ 2x && &+& 2z &=& 2, \\ 3x &+& y &+& 3z &=& 3. \end{array}$$

37. $$\begin{array}{rcrcrcr} 2x &+& y &+& z &=& 3, \\ 2x && &+& 2z &=& 2, \\ 4x &+& y &+& 3z &=& 5. \end{array}$$

38. $$\begin{array}{rcrcrcr} 2x &+& y &+& z &=& 2, \\ 6x && y &+& 5z &=& 2, \\ 4x &+& y &+& 3z &=& 2. \end{array}$$

39. $$\begin{array}{rcrcrcr} 6x &+& 2y &+& 6z &=& 10, \\ 6x && y &+& 6z &=& 11, \\ 4x &+& y &+& 4z &=& 7. \end{array}$$

40. $$\begin{array}{rcrcrcr} 6x &+& 2y &+& 4z &=& 6, \\ 6x && y &+& 5z &=& 9, \\ 4x &+& y &+& 3z &=& 5. \end{array}$$

Nonhomogeneous System. Solve the system using variable order y, z, u, v.

41. $$\begin{array}{rcrcrcrcr} y &+& z &+& 4u &+& 8v &=& 10, \\ && 2z &-& u &+& v &=& 10, \\ 2y && &-& u &+& 5v &=& 10. \end{array}$$

42. $$\begin{array}{rcrcrcrcr} y &+& z &+& 4u &+& 8v &=& 10, \\ && 2z &-& 2u &+& 2v &=& 0, \\ y &+& 3z &+& 2u &+& 5v &=& 5. \end{array}$$

43. $$\begin{array}{rcrcrcrcr} y &+& z &+& 4u &+& 8v &=& 1, \\ && 2z &-& 2u &+& 4v &=& 0, \\ y &+& 3z &+& 2u &+& 6v &=& 1. \end{array}$$

44. $$\begin{array}{rcrcrcrcr} y &+& 3z &+& 4u &+& 8v &=& 1, \\ && 2z &-& 2u &+& 4v &=& 0, \\ y &+& 3z &+& 2u &+& 6v &=& 1. \end{array}$$

45. $$\begin{array}{rcrcrcrcr} y &+& 3z &+& 4u &+& 8v &=& 1, \\ && 2z &-& 2u &+& 4v &=& 0, \\ y &+& 4z &+& 2u &+& 7v &=& 1. \end{array}$$

46. $$\begin{array}{rcrcrcrcr} y &+& z &+& 4u &+& 9v &=& 1, \\ && 2z &-& 2u &+& 4v &=& 0, \\ y &+& 4z &+& 2u &+& 7v &=& 1. \end{array}$$

47. $$\begin{array}{rcrcrcrcr} y &+& z &+& 4u &+& 9v &=& 1, \\ && 2z &-& 2u &+& 4v &=& 0, \\ y &+& 4z &+& 2u &+& 7v &=& 1. \end{array}$$

48. $$\begin{array}{rcrcrcrcr} y &+& z &+& 4u &+& 9v &=& 10, \\ && 2z &-& 2u &+& 4v &=& 4, \\ y &+& 4z &+& 2u &+& 7v &=& 8. \end{array}$$

49. $$\begin{array}{rcrcrcrcr} y &+& z &+& 4u &+& 9v &=& 2, \\ && 2z &-& 2u &+& 4v &=& 4, \\ y &+& 3z &+& 5u &+& 13v &=& 0. \end{array}$$

50. $$\begin{array}{rcrcrcrcr} y &+& z &+& 4u &+& 3v &=& 2, \\ && 2z &-& 2u &+& 4v &=& 4, \\ y &+& 3z &+& 5u &+& 7v &=& 0. \end{array}$$

3.4 Basis, Nullity and Rank

Studied here are the basic concepts of basis, nullity and rank of a system of linear algebraic equations.

Basis

Consider the homogeneous system

$$\begin{aligned} x + 2y + 3z &= 0, \\ 0 &= 0, \\ 0 &= 0. \end{aligned}$$

It is in reduced echelon form with standard general solution

$$\begin{aligned} &x = -2t_1 - 3t_2, \\ &y = t_1, \\ &z = t_2. \end{aligned}$$

The formal partial derivatives ∂_{t_1}, ∂_{t_2} of the general solution are solutions of the homogeneous system, because they correspond exactly to setting $t_1 = 1$, $t_2 = 0$ and $t_1 = 0$, $t_2 = 1$, respectively:

$$\begin{array}{llll} x = -2, & y = 1, & z = 0, & \text{(partial on } t_1) \\ x = -3, & y = 0, & z = 1. & \text{(partial on } t_2) \end{array}$$

The terminology **basis** is used to refer to the k homogeneous solutions obtained from the standard general solution by taking partial derivatives $\partial_{t_1}, \ldots, \partial_{t_k}$.

The general solution to the homogeneous system can be completely reconstructed from a basis, which motivates the terminology. In this sense, a *basis* is an abbreviation for the general solution.

Non-uniqueness of a Basis

A given linear system has a number of different standard general solutions, obtained, for example, by re-ordering the variable list. Therefore, a *basis is not unique.* Language like *the basis* is fundamentally incorrect.

To illustrate the idea, the homogeneous 3×3 system of equations

$$\begin{aligned} x + y + z &= 0, \\ 0 &= 0, \\ 0 &= 0, \end{aligned} \tag{19}$$

has two standard general solutions $x = -t_1 - t_2$, $y = t_1$, $z = t_2$ and $x = t_3$, $y = -t_3 - t_4$, $z = t_4$, corresponding to two different orderings of the variable list x, y, z. Then **two different bases** for the system are given by the partial derivative relations

$$\partial_{t_1}, \partial_{t_2} : \quad \begin{cases} x = -1, & y = 1, & z = 0, \\ x = -1, & y = 0, & z = 1, \end{cases} \tag{20}$$

$$\partial_{t_3}, \partial_{t_4} : \quad \begin{cases} x = 1, & y = -1, & z = 0, \\ x = 0, & y = -1, & z = 1. \end{cases} \tag{21}$$

Nullspace

The term **nullspace** refers to the set of all solutions of the homogeneous system. The prefix **null** refers to the right side of the system, which is zero, or *null*, for each equation. The word **space** has meaning taken from the phrases **storage space** and **parking space** — it has no intended geometrical meaning whatsoever.

Solutions of a homogeneous system are given by a general solution formula containing invented symbols t_1, t_2, ..., the meaning being that assignment of values to the symbols t_1, t_2, ... lists all possible solutions of the system.

A **basis** for the nullspace is found by taking partial derivatives ∂_{t_1}, ∂_{t_2}, ... on the general solution, giving k solutions. The general solution is reconstructed from these basis elements by multiplying them by the symbols t_1, t_2, ... and adding. The nullspace is the same regardless of the choice of basis, it is just the set of solutions of the homogeneous equation.

For the preceding illustration (19), the nullspace is the set of all solutions of $x + y + z = 0$. Geometrically, it is the plane $x + y + z = 0$ through $x = y = z = 0$ with normal vector $\vec{\imath} + \vec{\jmath} + \vec{k}$. The nullspace is represented via either basis (20) or (21) by combinations of basis elements. For instance, the first basis (20) gives the general solution formula

$$\begin{aligned} x &= -t_1 - t_2, \\ y &= t_1, \\ z &= t_2. \end{aligned}$$

Rank, Nullity and Dimension

The **rank** of a system of equations is defined to be the **number of lead variables** in an equivalent reduced row echelon system. The **nullity** of a system of equations is the **number of free variables** appearing in an equivalent reduced echelon system.

The nullity is exactly the number k of partial derivatives taken to compute the elements in a basis for the nullspace. For instance, the nullity of system (19) equals 2 because there are two free variables, which were assigned the invented symbols t_1, t_2.

In literature, nullity is referred to as the **dimension** of the nullspace. The term *dimension* is a synonym for the **number of free variables**, which is exactly the **number of parameters** in a standard general solution for the linear system, or equivalently, the **number of partial derivatives** taken to compute a basis.

The fundamental relations between rank and nullity are

$$\begin{aligned}&\textbf{rank} = \textbf{number of lead variables},\\&\textbf{nullity} = \textbf{number of free variables},\\&\textbf{rank} + \textbf{nullity} = \textbf{number of variables}.\end{aligned}$$

The Three Possibilities, Rank and Nullity

We intend to justify the table below, which summarizes the three possibilities for a linear system, in terms of free variables, rank and nullity.

Table 5. Three possibilities for a linear system.

Unique solution	Zero free variables	nullity $= 0$
No solution	Signal equation $0 = 1$	
Infinitely many solutions	One or more free variables	nullity ≥ 1

Unique Solution. There is a unique solution to a system of equations exactly when **zero free variables** are present. This is identical to requiring that the number n of variables equal the number of lead variables, or $\mathbf{rank = n}$.

No Solution. There is no solution to a system of equations exactly when a signal equation $0 = 1$ occurs during the application of swap, multiply and combination rules. We report the system **inconsistent** and announce **no solution**.

Infinitely Many Solutions. The situation of infinitely many solutions occurs when there is **at least one free variable** to which an invented symbol, say t_1, is assigned. Since this symbol takes the values $-\infty < t_1 < \infty$, there are an infinity of solutions. The condition **rank less than n** can replace a reference to the number of free variables.

Homogeneous systems are always consistent, therefore if the number of variables exceeds the number of equations, then there is always one free variable. This proves the following basic result of linear algebra.

Theorem 5 (Existence of Infinitely Many Solutions)
A system of $m \times n$ linear homogeneous equations (6) with fewer equations than unknowns ($m < n$) has at least one free variable, hence an infinite number of solutions. Therefore, such a system always has the **zero solution** and also a **nonzero solution**.

Examples and Methods

9 Example (Three Possibilities with Symbol k) Determine all values of the symbol k such that the system below has (1) No solution, (2) Infinitely many solutions or (3) A unique solution. Display all solutions found.

$$\begin{array}{rcrcl} x & + & ky & = & 2, \\ (2-k)x & + & y & = & 3. \end{array}$$

Solution: The solution of this problem involves construction of three frame sequences, the last frame of each resulting in one classification among the Three Possibilities: (1) No solution, (2) Unique solution, (3) Infinitely many solutions.

The plan, for each of the three possibilities, is to obtain a triangular system by application of swap, multiply and combination rules. Each step tries to increase the number of leading variables. The three possibilities are detected by (1) A signal equation "$0 = 1$," (2) One or more free variables, (3) Zero free variables. A portion of the frame sequence is constructed, as follows.

$$\boxed{\begin{array}{rcrcr} x & + & ky & = & 2, \\ (2-k)x & + & y & = & 3. \end{array}}$$

Frame 1.

Original system.

$$\boxed{\begin{array}{rcrcr} x & + & ky & = & 2, \\ 0 & + & [1+k(k-2)]y & = & 2(k-2)+3. \end{array}}$$

Frame 2.

`combo(1,2,k-2)`

$$\boxed{\begin{array}{rcrcr} x & + & ky & = & 2, \\ 0 & + & (k-1)^2 y & = & 2k-1. \end{array}}$$

Frame 3.

Simplify.

The three expected frame sequences share these initial frames. At this point, we identify the values of k that split off into the three possibilities.

There will be a signal equation if the second equation of Frame 3 has no variables, but the resulting equation is not "$0 = 0$." This happens exactly for $k = 1$. The resulting signal equation is "$0 = 1$." We conclude that one of the three frame sequences terminates with the *no solution case*. This frame sequence corresponds to $k = 1$.

Otherwise, $k \neq 1$. For these values of k, there are zero free variables, which implies a unique solution. A by-product of the analysis is that the *infinitely many solutions* case never occurs!

The conclusion: the three frame sequences reduce to two frame sequences. One sequence gives no solution and the other sequence gives a unique solution.

The three answers:

(1) There is no solution only for $k = 1$.

(2) Infinitely many solutions never occur for any value of k.

(3) For $k \neq 1$, there is a unique solution

$$x = 2 - k(2k-1)/(k-1)^2,$$
$$y = (2k-1)/(k-1)^2.$$

10 Example (Symbols and the Three Possibilities) Determine all values of the symbols a, b such that the system below has (1) No solution, (2) Infinitely many solutions or (3) A unique solution. Display all solutions found.

$$\begin{array}{rcrcrcl} x & + & ay & + & bz & = & 2, \\ & & y & + & z & = & 3, \\ & & by & + & z & = & 3b. \end{array}$$

Solution: The plan is to make three frame sequences, using swap, multiply and combination rules. Each sequence has last frame which is one of the three possibilities, the detection facilitated by (1) A signal equation "$0 = 1$," (2) At least one free variable, (3) Zero free variables. The initial portion of each frame sequence is constructed as follows.

$\begin{array}{rcrcrcr} x & + & ay & + & bz & = & 2, \\ & & y & + & z & = & 3, \\ & & by & + & z & = & 3b. \end{array}$	Frame 1 Original system.
$\begin{array}{rcrcrcr} x & + & ay & + & bz & = & 2, \\ & & y & + & z & = & 3, \\ & & 0 & + & (1-b)z & = & 0. \end{array}$	Frame 2. `combo(2,3,-b)`
$\begin{array}{rcrcrcr} x & + & 0 & + & (b-a)z & = & 2-3a, \\ & & y & + & z & = & 3, \\ & & 0 & + & (1-b)z & = & 0. \end{array}$	Frame 3. `combo(2,1,-a)` Triangular form. Lead variables determined.

The three frame sequences expected will share these initial frames. Frame 3 shows that there are either 2 lead variables or 3 lead variables, accordingly as the coefficient of z in the third equation is nonzero or zero. There will never be a signal equation. Consequently, the three expected frame sequences reduce to just two. We complete these two sequences to give the answer:

(1) There are no values of a, b that result in no solution.

(2) If $1 - b = 0$, then there are two lead variables and hence an infinite number of solutions, given by the general solution

$$\begin{cases} x = 2 - 3a - (b-a)t_1, \\ y = 3 - t_1, \\ z = t_1. \end{cases}$$

(3) If $1 - b \neq 0$, then there are three lead variables and there is a unique solution, given by

$$\begin{cases} x = 2 - 3a, \\ y = 3, \\ z = 0. \end{cases}$$

Exercises 3.4

Nullspace. Solve using variable order y, z, u, v. Report the values of the **nullity** and **rank** in the equation **nullity+rank**=4.

1. $$\begin{array}{rcrcrcrl} y &+& z &+& 4u &+& 8v &= 0,\\ && 2z &-& u &+& v &= 0,\\ 2y &&&-& u &+& 5v &= 0.\end{array}$$

2. $$\begin{array}{rcrcrcrl} y &+& z &+& 4u &+& 8v &= 0,\\ && 2z &-& 2u &+& 2v &= 0,\\ y &+& 3z &+& 2u &+& 5v &= 0.\end{array}$$

3. $$\begin{array}{rcrcrcrl} y &+& z &+& 4u &+& 8v &= 0,\\ && 2z &-& 2u &+& 4v &= 0,\\ y &+& 3z &+& 2u &+& 6v &= 0.\end{array}$$

4. $$\begin{array}{rcrcrcrl} y &+& 3z &+& 4u &+& 8v &= 0,\\ && 2z &-& 2u &+& 4v &= 0,\\ y &+& 3z &+& 2u &+& 6v &= 0.\end{array}$$

5. $$\begin{array}{rcrcrcrl} y &+& 3z &+& 4u &+& 8v &= 0,\\ && 2z &-& 2u &+& 4v &= 0.\end{array}$$

6. $$\begin{array}{rcrcrcrl} y &+& z &+& 4u &+& 9v &= 0,\\ && 2z &-& 2u &+& 4v &= 0.\end{array}$$

7. $$\begin{array}{rcrcrcrl} y &+& z &+& 4u &+& 9v &= 0,\\ 3y &+& 4z &+& 2u &+& 5v &= 0.\end{array}$$

8. $$\begin{array}{rcrcrcrl} y &+& 2z &+& 4u &+& 9v &= 0,\\ y &+& 8z &+& 2u &+& 7v &= 0.\end{array}$$

9. $$\begin{array}{rcrcrcrl} y &+& z &+& 4u &+& 11v &= 0,\\ && 2z &-& 2u &+& 4v &= 0.\end{array}$$

10. $$\begin{array}{rcrcrcrl} y &+& z &+& 5u &+& 11v &= 0,\\ && 2z &-& 2u &+& 6v &= 0.\end{array}$$

RREF. In the homogeneous systems, assume variable order x, y, z, u, v. (a) Display an equivalent set of equations in reduced echelon form (**rref**), (b) Solve for the general solution, (c) Check the answer.

11. $$\begin{array}{rcrcrcrcrl} x &+& y &+& z &+& 4u &+& 8v &= 0,\\ -x &+& && 2z &-& 2u &+& 2v &= 0,\\ && y &-& z &+& 6u &+& 6v &= 0.\end{array}$$

12. $$\begin{array}{rcrcrcrcrl} x &+& y &+& z &+& 4u &+& 8v &= 0,\\ &&&-& 2z &-& u &+& v &= 0,\\ && 2y &&&-& u &+& 5v &= 0.\end{array}$$

13. $$\begin{array}{rcrcrcrcrl} && y &+& z &+& 4u &+& 8v &= 0,\\ x &&&+& 2z &-& 2u &+& 4v &= 0,\\ 2x &+& y &+& 3z &+& 2u &+& 6v &= 0.\end{array}$$

14. $$\begin{array}{rcrcrcrcrl} x &+& y &+& 3z &+& 4u &+& 8v &= 0,\\ 2x &&&+& 2z &-& 2u &+& 4v &= 0,\\ x &-& y &+& 3z &+& 2u &+& 12v &= 0.\end{array}$$

15. $$\begin{array}{rcrcrcrl} y &+& 3z &+& 4u &+& 20v &= 0,\\ &+& 2z &-& 2u &+& 10v &= 0,\\ -\ y &+& 3z &+& 2u &+& 30v &= 0.\end{array}$$

16. $$\begin{array}{rcrcrl} y &+& 4u &+& 20v &= 0,\\ &-& 2u &+& 10v &= 0,\\ -\ y &+& 2u &+& 30v &= 0.\end{array}$$

17. $$\begin{array}{rcrcrcrcl} x &+& y &+& z &+& 4u && = 0,\\ &&&-& 2z &-& u && = 0,\\ && 2y &&&-& u+ && = 0.\end{array}$$

18. $$\begin{array}{rcrcrcrl} &+& z &+& 12u &+& 8v &= 0,\\ x &+& 2z &-& 6u &+& 4v &= 0,\\ 2x &+& 3z &+& 6u &+& 6v &= 0.\end{array}$$

19. $$\begin{array}{rcrcrl} y &+& z &+& 4u &= 0,\\ && 2z &-& 2u &= 0,\\ y &-& z &+& 6u &= 0.\end{array}$$

20. $$\begin{array}{rcrcrl} x &+& z &+& 8v &= 0,\\ &-& 2z &+& v &= 0,\\ &&&& 5v &= 0.\end{array}$$

Three possibilities. Assume variables x, y, z. Determine the values of the constants (a, b, c, k, etc) such that the system has (1) *No solution*, (2) *Infinitely many solutions* or (3) *A unique solution.*

21. $$\begin{array}{rcrl} x &+& ky &= 0,\\ x &+& 2ky &= 0.\end{array}$$

22. $$\begin{array}{rcrl} kx &+& ky &= 0,\\ x &+& 2ky &= 0.\end{array}$$

23. $$\begin{array}{rcrl} ax &+& by &= 0,\\ x &+& 2by &= 0.\end{array}$$

24. $$\begin{aligned} bx + ay &= 0, \\ x + 2y &= 0. \end{aligned}$$

25. $$\begin{aligned} bx + ay &= c, \\ x + 2y &= b - c. \end{aligned}$$

26. $$\begin{aligned} bx + ay &= 2c, \\ x + 2y &= c + a. \end{aligned}$$

27. $$\begin{aligned} bx + ay + z &= 0, \\ 2bx + ay + 2z &= 0, \\ x + 2y + 2z &= c. \end{aligned}$$

28. $$\begin{aligned} bx + ay + z &= 0, \\ 3bx + 2ay + 2z &= 2c, \\ x + 2y + 2z &= c. \end{aligned}$$

29. $$\begin{aligned} 3x + ay + z &= b, \\ 2bx + ay + 2z &= 0, \\ x + 2y + 2z &= c. \end{aligned}$$

30. $$\begin{aligned} x + ay + z &= 2b, \\ 3bx + 2ay + 2z &= 2c, \\ x + 2y + 2z &= c. \end{aligned}$$

Three Possibilities. The following questions can be answered by using the quantitative expression of the three possibilities in terms of lead and free variables, rank and nullity.

31. Does there exist a homogeneous 3×2 system with a unique solution? Either give an example or else prove that no such system exists.

32. Does there exist a homogeneous 2×3 system with a unique solution? Either give an example or else prove that no such system exists.

33. In a homogeneous 10×10 system, two equations are identical. Prove that the system has a nonzero solution.

34. In a homogeneous 5×5 system, each equation has a leading variable. Prove that the system has only the zero solution.

35. Suppose given two homogeneous systems A and B, with A having a unique solution and B having infinitely many solutions. Explain why B cannot be obtained from A by a sequence of swap, multiply and combination operations on the equations.

36. A 2×3 system cannot have a unique solution. Cite a theorem or explain why.

37. If a 3×3 homogeneous system contains no variables, then what is the general solution?

38. If a 3×3 non-homogeneous solution has a unique solution, then what is the nullity of the homogeneous system?

39. A 7×7 homogeneous system is missing two variables. What is the maximum rank of the system? Give examples for all possible ranks.

40. Suppose an $n \times n$ system of equations (homogeneous or nonhomogeneous) has two solutions. Prove that it has infinitely many solutions.

41. What is the nullity and rank of an $n \times n$ system of homogeneous equations if the system has a unique solution?

42. What is the nullity and rank of an $n \times n$ system of non-homogeneous equations if the system has a unique solution?

43. Prove or disprove (by example): A 4×3 nonhomogeneous system cannot have a unique solution.

44. Prove or disprove (by example): A 4×3 homogeneous system always has infinitely many solutions.

3.5 Answer Check, Proofs and Details

Answer Check Algorithm

A given general solution (12) can be tested for validity manually as in Example 4, page 181. It is possible to devise a **symbol-free answer check**. The technique checks a general solution (12) by testing constant trial solutions in systems (5) and (6).

Step 1. Set all symbols to zero in general solution (12) to obtain the nonhomogeneous trial solution $x_1 = d_1$, $x_2 = d_2$, ..., $x_n = d_n$. Test it by direct substitution into the nonhomogeneous system (5).

Step 2. Apply partial derivatives ∂_{t_1}, ∂_{t_2}, ..., ∂_{t_k} to the general solution (12), obtaining k homogeneous trial solutions. Verify that the trial solutions satisfy the homogeneous system (6), by direct substitution.

The trial solutions in **step 2** are obtained from the general solution (12) by setting one symbol equal to 1 and the others zero, followed by subtracting the nonhomogeneous trial solution of **step 1**. The partial derivative idea computes the same set of trial solutions, and it is easier to remember.

Theorem 6 (Answer Check)
The answer check algorithm described in steps 1–2 verifies the general solution (12) for all values of the symbols. Please observe that this answer check cannot test for skipped solutions.

Proof of Theorem 6. To simplify notation and quickly communicate the ideas, a proof will be given for a 2×2 system. A proof for the $m \times n$ case can be constructed by the reader, using the same ideas. Consider the nonhomogeneous and homogeneous systems

$$\begin{aligned} ax_1 + by_1 &= b_1, \\ cx_1 + dy_1 &= b_2, \end{aligned} \tag{22}$$

$$\begin{aligned} ax_2 + by_2 &= 0, \\ cx_2 + dy_2 &= 0. \end{aligned} \tag{23}$$

Assume (x_1, y_1) is a solution of (22) and (x_2, y_2) is a solution of (23). Add corresponding equations in (22) and (23). Then collecting terms gives

$$\begin{aligned} a(x_1 + x_2) + b(y_1 + y_2) &= b_1, \\ c(x_1 + x_2) + d(y_1 + y_2) &= b_2. \end{aligned} \tag{24}$$

This proves that $(x_1 + x_2, y_1 + y_2)$ is a solution of the nonhomogeneous system. Similarly, a scalar multiple (kx_2, ky_2) of a solution (x_2, y_2) of system (23) is also a solution of (23) and the sum of two solutions of (23) is again a solution of (23).

Given each solution in **step 2** satisfies (23), then multiplying the first solution by t_1 and the second solution by t_2 and adding gives a solution (x_3, y_3) of (23). After adding (x_3, y_3) to the solution (x_1, y_1) of **step 1**, a solution of (22) is obtained, proving that the full parametric solution containing the symbols t_1, t_2 is a solution of (22). The proof for the 2×2 case is complete.

Failure of Answer Checks

An answer check only tests the given formulas against the equations. If too few parameters are present, then the answer check can be algebraically correct but the general solution check fails, because not all solutions can be obtained by specialization of the parameter values.

For example, $x = 1 - t_1$, $y = t_1$, $z = 0$ is a one-parameter solution for $x + y + z = 1$, as verified by an answer check. But the general solution $x = 1 - t_1 - t_2$, $y = t_1$, $z = t_2$ has two parameters t_1, t_2. Generally, an answer check decides if the formula supplied works in the equation. It does **not** decide if the given formula represents **all** solutions. This trouble, in which an error leads to a *smaller* value for the nullity of the system, is due largely to human error and not machine error.

Linear algebra workbenches have another kind of flaw: they may compute the **nullity** for a system incorrectly as an integer *larger* than the correct nullity. A parametric solution with nullity k might be obtained, checked to work in the original equations, then cross-checked by computing the nullity k independently. However, the computed nullity k could be greater than the actual nullity of the system. Here is a simple example, where ϵ is a *very* small positive number:

$$\begin{array}{rcrcl} x & + & y & = & 0, \\ & & \epsilon y & = & \epsilon. \end{array} \tag{25}$$

On a limited precision machine, system (25) has internal machine representation[2]

$$\begin{array}{rcrcl} x & + & y & = & 0, \\ & & 0 & = & 0. \end{array} \tag{26}$$

Representation (26) occurs because the coefficient ϵ is smaller than the smallest positive floating point number of the machine, hence it becomes zero during translation. System (25) has nullity zero and system (26) has nullity one. The parametric solution for system (26) is $x = -t_1$, $y = t_1$, with basis selected by setting $t_1 = 1$. The basis passes the answer check on system (25), because ϵ times 1 evaluates to ϵ. A second check

[2]For example, if the machine allows only 2-digit exponents (10^{99} is the maximum), then $\epsilon = 10^{-101}$ translates to zero.

for the nullity of system (26) gives 1, which supports the correctness of the parametric solution, but unfortunately there are not infinitely many solutions: for system (25) the correct answer is the unique solution $x = -1$, $y = 1$.

Computer algebra systems (CAS) are supposed to avoid this kind of error, because they do not translate input into floating point representations. All input is supposed to remain in symbolic or in string form. In short, they don't change ϵ to zero. Because of this standard, CAS are safer systems in which to do linear algebra computations, albeit slower in execution.

The trouble reported here is not entirely one of input translation. An innocuous `combo(1,2,-1)` can cause an equation like $\epsilon y = \epsilon$ in the middle of a frame sequence. If floating point hardware is being used, and not symbolic computation, then the equation can translate to $0 = 0$, causing a false free variable appearance.

Minimal Parametric Solutions

Proof of Theorem 2: The proof of Theorem 2, page 176, will follow from the lemma and theorem below.

Lemma 1 (Unique Representation) If a set of parametric equations (12) satisfies (13), (14) and (15), then each solution of linear system (5) is given by (12) for exactly one set of parameter values.

Proof: Let a solution of system (5) be given by (12) for two sets of parameters $t_1, \ldots, t_k$ and $\bar{t}_1, \ldots, \bar{t}_k$. By (15), $t_j = x_{i_j} = \bar{t}_j$ for $1 \le j \le k$, therefore the parameter values are the same.

Definition 5 (Minimal Parametric Solution)
Given system (5) has a parametric solution $x_1, \ldots, x_n$ satisfying (12), (13), (14), then among all such parametric solutions there is one which uses the *fewest* possible parameters. A parametric solution with fewest parameters is called **minimal**. Parametric solutions with more parameters are called **redundant**.

To illustrate, the plane $x+y+z = 1$ has a minimal standard parametric solution $x = 1-t_1-t_2$, $y = t_1$, $z = t_2$. A redundant parametric solution of $x+y+z = 1$ is $x = 1 - t_1 - t_2 - 2t_3$, $y = t_1 + t_3$, $z = t_2 + t_3$, using three parameters t_1, t_2, t_3.

Theorem 7 (Minimal Parametric Solutions)
Let linear system (5) have a parametric solution satisfying (12), (13), (14). Then (12) has the fewest possible parameters if and only if each solution of linear system (5) is given by (12) for exactly one set of parameter values.

Proof: Suppose first that a general solution (12) is given with the least number k of parameters, but contrary to the theorem, there are two ways to represent some solution, with corresponding parameters $r_1, \ldots, r_k$ and also $s_1, \ldots, s_k$. Subtract the two sets of parametric equations, thus eliminating the symbols $x_1, \ldots, x_n$, to obtain:

$$\begin{aligned} c_{11}(r_1 - s_1) + \cdots + c_{1k}(r_k - s_k) &= 0, \\ &\vdots \\ c_{n1}(r_1 - s_1) + \cdots + c_{nk}(r_k - s_k) &= 0. \end{aligned}$$

Relabel the variables and constants so that $r_1 - s_1 \neq 0$, possible since the two sets of parameters are supposed to be different. Divide the preceding equations by $r_1 - s_1$ and solve for the constants $c_{11}, \ldots, c_{n1}$. This results in equations

$$\begin{aligned} c_{11} &= c_{12}w_2 + \cdots + c_{1k}w_k, \\ &\vdots \\ c_{n1} &= c_{n2}w_2 + \cdots + c_{nk}w_k, \end{aligned}$$

where $w_j = -\frac{r_j - s_j}{r_1 - s_1}$, $2 \le j \le k$. Insert these relations into (12), effectively eliminating the symbols $c_{11}, \ldots, c_{n1}$, to obtain

$$\begin{aligned} x_1 &= d_1 + c_{12}(t_2 + w_2t_1) + \cdots + c_{1k}(t_k + w_kt_1), \\ x_2 &= d_2 + c_{22}(t_2 + w_2t_1) + \cdots + c_{2k}(t_k + w_kt_1), \\ &\vdots \\ x_n &= d_n + c_{n2}(t_2 + w_2t_1) + \cdots + c_{nk}(t_k + w_kt_1). \end{aligned}$$

Let $t_1 = 0$. The remaining parameters $t_2, \ldots, t_k$ are fewer parameters that describe all solutions of the system, a contradiction to the definition of k. This completes the proof of the first half of the theorem.

To prove the second half of the theorem, assume that a parametric solution (12) is given which represents all possible solutions of the system and in addition each solution is represented by exactly one set of parameter values. It will be established that the number k in (12) is the *least possible* parameter count.

Suppose not. Then there is a second parametric solution

$$\begin{aligned} x_1 &= e_1 + b_{11}v_1 + \cdots + b_{1\ell}v_\ell, \\ &\vdots \\ x_n &= e_n + b_{n1}v_1 + \cdots + b_{n\ell}v_\ell, \end{aligned} \tag{27}$$

where $\ell < k$ and $v_1, \ldots, v_\ell$ are the parameters. It is assumed that (27) represents all solutions of the linear system.

We shall prove that the solutions for zero parameters in (12) and (27) can be taken to be the same, that is, another parametric solution is given by

$$\begin{aligned} x_1 &= d_1 + b_{11}s_1 + \cdots + b_{1\ell}s_\ell, \\ &\vdots \\ x_n &= d_n + b_{n1}s_1 + \cdots + b_{n\ell}s_\ell. \end{aligned} \tag{28}$$

The idea of the proof is to substitute $x_1 = d_1, \ldots, x_n = d_n$ into (27) for parameters $r_1, \ldots, r_n$. Then solve for $e_1, \ldots, e_n$ and replace back into (27) to obtain

$$\begin{aligned} x_1 &= d_1 + b_{11}(v_1 - r_1) + \cdots + b_{1\ell}(v_\ell - r_\ell), \\ &\vdots \\ x_n &= d_n + b_{n1}(v_1 - r_1) + \cdots + b_{n\ell}(v_\ell - r_\ell). \end{aligned}$$

Replacing parameters $s_j = v_j - r_j$ gives (28).

From (12) it is known that $x_1 = d_1 + c_{11}, \ldots, x_n = d_n + c_{n1}$ is a solution. By (28), there are constants $r_1, \ldots, r_\ell$ such that (we cancel $d_1, \ldots, d_n$ from both sides)

$$\begin{aligned} c_{11} &= b_{11}r_1 + \cdots + b_{1\ell}r_\ell, \\ &\vdots \\ c_{n1} &= b_{n1}r_1 + \cdots + b_{n\ell}r_\ell. \end{aligned}$$

If r_1 through r_ℓ are all zero, then the solution just referenced equals $d_1, \ldots, d_n$, hence (12) has a solution that can be represented with parameters all zero or with $t_1 = 1$ and all other parameters zero, a contradiction. Therefore, some $r_i \neq 0$ and we can assume by renumbering that $r_1 \neq 0$. Return now to the last system of equations and divide by r_1 in order to solve for the constants $b_{11}, \ldots, b_{n1}$. Substitute the answers back into (28) in order to obtain parametric equations

$$\begin{aligned} x_1 &= d_1 + c_{11}w_1 + b_{12}w_2 + \cdots + b_{1\ell}w_\ell, \\ &\vdots \\ x_n &= d_n + c_{n1}w_1 + b_{n2}w_2 + \cdots + b_{n\ell}w_\ell, \end{aligned}$$

where $w_1 = s_1$, $w_j = s_j - r_j/r_1$. Given $s_1, \ldots, s_\ell$ are parameters, then so are $w_1, \ldots, w_\ell$.

This process can be repeated for the solution $x_1 = d_1 + c_{12}, \ldots, x_n = d_n + c_{n2}$. We assert that for some index j, $2 \le j \le \ell$, constants $b_{ij}, \ldots, b_{nj}$ in the previous display can be isolated, and the process of replacing symbols b by c continued. If not, then $w_2 = \cdots = w_\ell = 0$. Then solution $x_1, \ldots, x_n$ has two distinct representations in (12), first with $t_2 = 1$ and all other $t_j = 0$, then with $t_1 = w_1$ and all other $t_j = 0$. A contradiction results, which proves the assertion. After ℓ repetitions of this replacement process, we find a parametric solution

$$\begin{aligned} x_1 &= d_1 + c_{11}u_1 + c_{12}u_2 + \cdots + c_{1\ell}u_\ell, \\ &\vdots \\ x_n &= d_n + c_{n1}u_1 + c_{n2}u_2 + \cdots + c_{n\ell}u_\ell, \end{aligned}$$

in some set of parameters $u_1, \ldots, u_\ell$.

However, $\ell < k$, so at least the solution $x_1 = d_1 + c_{1k}, \ldots, x_n = d_n + c_{nk}$ remains unused by the process. Insert this solution into the previous display, valid for some parameters $u_1, \ldots, u_\ell$. The relation says that the solution $x_1 = d_1, \ldots, x_n = d_n$ in (12) has two distinct sets of parameters, namely $t_1 = u_1, \ldots, t_\ell = u_\ell$, $t_k = -1$, all others zero, and also all parameters zero, a contradiction. This completes the proof of the theorem.