Introduction to Linear Algebra 2270-2
Sample Midterm Exam 3 Spring 2007
Exam Date: Wednesday, 18 April 2007

Instructions. The exam is 50 minutes. Calculators are not allowed. Books and notes are not allowed. More choices appear on the sample exam than will appear on exam day.

1. (Kernel, Independence, Similarity) Complete two.

(a) Use the identity $\mathbf{rref}(A) = E_1 E_2 \cdots E_k A$ to prove: $\ker(A) = \{\mathbf{0}\}$ if and only if $\det(A) \neq 0$.

(b) Assume $n \times n$ matrix A satisfies $A^k \neq 0$ and $A^k A = 0$ for some integer $k \geq 0$. Choose $\mathbf{v}$ with $A^k \mathbf{v} \neq \mathbf{0}$. Prove (1) and (2):
(1) Vectors $\mathbf{v}$, $A\mathbf{v}$, $A^2\mathbf{v}$, $\ldots$, $A^k\mathbf{v}$ are linearly independent.
(2) Always, $k < n$. Hence $A^n = 0$.

(c) Suppose for matrices A, B the product AB is defined. Prove that $\ker(A) = \ker(B) = \{\mathbf{0}\}$ implies $\ker(AB) = \{\mathbf{0}\}$.

(d) Do there exist matrices A and B such that A is not similar to B but $A - 2I$ is similar to $B - 2I$? Justify.

Please start your solutions on this page. Additional pages may be stapled to this one.

2. (Abstract vector spaces, Linear transformations) Complete two.

Let W be the set of all infinite sequences of real numbers $\mathbf{x} = \{x_n\}_{n=0}^{\infty}$ (Section 4.1, page 154).

(a) Define addition and scalar multiplication for W and prove that W is a vector space.

(b) Let V be the subset of W defined by $\sum_{n=0}^{\infty} |x_n|^2 < \infty$. Prove that V is a subspace of W.

(c) Define $T(\mathbf{x}) = \{x_{n+1}\}_{n=0}^{\infty}$ on V. Show that T is a linear transformation from V to V and determine $\ker(T)$.

(d) Define $S(f) = 2f - f'$ from $X = C^{\infty}[0,1]$ into X. Find the kernel and nullity of S.

Please start your solutions on this page. Additional pages may be stapled to this one.

3. (Orthogonality, Gram-Schmidt) Complete two.

(a) Give an algebraic proof, depending only on inner product space properties, of the triangle inequality $\|\mathbf{u}+\mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|$ in $\mathcal{R}^n$.

(b) Find the orthogonal projection of $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ onto $V = \textbf{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \\ -1 \end{pmatrix} \right\}$.

(c) Find the QR-factorization of $A = \begin{pmatrix} 1 & 0 & 1 \\ 7 & 7 & 8 \\ 1 & 2 & 1 \\ 7 & 7 & 6 \end{pmatrix}$.

(d) Find the QR-factorization of $A = \begin{pmatrix} 4 & 25 & 0 \\ 0 & 0 & -2 \\ 3 & -25 & 0 \end{pmatrix}$.

(e) Give 5 equivalent statements for an $n \times n$ matrix A to be orthogonal.

(f) Prove that an invertible matrix A has exactly one QR-factorization.

Please start your solutions on this page. Additional pages may be stapled to this one.

4. (Orthogonality and least squares) Complete two.

(a) Prove that $\ker(A) = \ker(A^T A)$ and that $A^T A$ is invertible when $\ker(A) = \{\mathbf{0}\}$.

(b) For an inconsistent system $A\mathbf{x} = \mathbf{b}$, the least squares solutions $\mathbf{x}$ are the exact solutions of the normal equation. Define the normal equation and display the unique solution $\mathbf{x} = \mathbf{x}^*$ when $\ker(A) = \{\mathbf{0}\}$.

(c) Prove the *near point theorem*: Given a vector $\mathbf{x}$ in $\mathcal{R}^n$ and a subspace V of $\mathcal{R}^n$, then $\mathbf{v} = \mathbf{proj}_V(\mathbf{x})$ is the nearest point in V to $\mathbf{x}$. This statement means that the minimum of $\|\mathbf{x} - \mathbf{v}\|$ is attained over all $\mathbf{v}$ in V at precisely the one point $\mathbf{v} = \mathbf{proj}_V(\mathbf{x})$.

(d) Fit $c_0 + c_1 x + c_2 x^2$ to the data points $(0, 27)$, $(1, 0)$, $(2, 0)$, $(3, 0)$ using least squares. Sketch the solution and the data points as an answer check.

Please start your solutions on this page. Additional pages may be stapled to this one.

5. (Determinants) Complete two.

(a) Given a 7×7 matrix A with each entry either a zero or a one, then what is the least number of zero entries possible such that A is invertible?

(b) Find A^{-1} by two methods: the classical adjoint method and the **rref** method applied to $\mathbf{aug}(A, I)$:

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}.$$

(c) Let 4×4 matrix A be invertible and assume $\mathbf{rref}(A) = E_3 E_2 E_2 A$. The elementary matrices E_1, E_2, E_3 represent `combo(1,3,-15)`, `swap(1,4)`, `mult(2,-1/4)`, respectively. Find $\det(A)$.

(d) Let $C + B^2 + BA = A^2 + AB$. Assume $\det(A - B) = 4$ and $\det(C) = 5$. Find $\det(CA + CB)$.