KEY

Introduction to Linear Algebra 2270-2
Midterm Exam 3 Spring 2007
Exam Date: Wednesday, 18 April 2007

Instructions. The exam is 50 minutes. Calculators are not allowed. Books and notes are not allowed.

1. **(Kernel, Independence, Similarity)** Complete two.

(a) [50%] Use the identity $\mathbf{rref}(A) = E_1 E_2 \cdots E_k A$ to prove: If $\mathbf{ker}(A) = \{\mathbf{0}\}$, then $\det(A) = \dfrac{(-1)^q}{m_1 \cdots m_p}$, where q is the number of elementary swap matrices and $m_1, \ldots, m_p$ are the multipliers for the elementary multiply matrices, in the sequence $E_1, \ldots, E_k$.

(b) [50%] Suppose the matrices A and B are 3×3. Prove or disprove:

$$\mathbf{ker}(AB) = \{\mathbf{0}\} \text{ implies } \mathbf{ker}(BA) = \{\mathbf{0}\}.$$

(c) [50%] If you did (a) and (b), then stop, because 100% has been obtained. Otherwise continue. Do there exist matrices A and B such that A is not similar to B but A^3 is similar to B^3? Justify. Hint: This problem comes from maple lab problem L2.3.

(a) Because $\ker(A) = \{\vec{0}\} \Rightarrow \det(A) \neq 0$ and $\mathrm{rref}(A) = I$, then $I = E_1 \cdots E_k A$ and $\det(I) = \det(E_1 \cdots E_k A)$. The product theorem for determinants applies to give $1 = \det(E_1) \cdots \det(E_k) \det(A)$. Because $\det(E) = -1, 1$ or m for swap, combo, mult(j, m), resp., then $\det(A) = \dfrac{1}{(-1)^q\, m_1 \cdots m_p}$. The proof is complete.

(b) prove first: $\ker(B) = \{0\}$. Choose $x \in \ker(B)$. Then $Bx = 0$. multiply by A to get $ABx = 0$. Then $x = 0$, because $\ker(AB) = \{0\}$.

prove $\ker(BA) = \{0\}$. Choose x s.t. $BAx = 0$. multiply by A to get $ABAx = 0$. Then $Ax = 0$, because $\ker(AB) = \{0\}$.

Define $y = B^{-1}x$. Then

$$0 = Ax = ABy$$

implies $y = 0$, because $\ker(AB) = \{0\}$. Then $x = By = 0$. proof complete

(c) we choose A, B such that $A^3 = B^3 = 0$. Then A^3, B^3 are similar. Let $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. If $AP = PB$, then $\mathrm{col}(AP, 1) = \mathrm{col}(AP, 2) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. But cols of P are independent, so $A\vec{x} = \vec{0}$ has 2 indep solutions. It does not — all solutions are $\vec{x} = t_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

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2. (Abstract vector spaces, Linear transformations) Complete two.

Let W be the set of all infinite sequences of real numbers $\mathbf{x} = \{x_n\}_{n=0}^{\infty}$. Define addition and scalar multiplication for W by the usual rules in the textbook. Assume W is known to be a vector space.

(a) [50%] Let V be the subset of W defined by the relation $\lim_{n\to\infty} x_n = 0$. Prove that V is a subspace of W.

(b) [50%] Let V be defined as in (a) above. Define $T(\mathbf{x}) = \{y_n\}_{n=0}^{\infty}$ on V by the relations $y_0 = 0$, $y_1 = 0$, $y_{n+2} = x_n$ for $n \geq 0$. Show that T is a linear transformation from V to V.

(c) [50%] If you did (a) and (b), then stop with 100%. Otherwise continue.
Define T as in (b) above. Determine the kernel of T.

(a) Zero $\vec{0}$ is the zero sequence, and it has limit $= 0$, so $\vec{0}$ is in V. Given scalars c_1, c_2 and sequences $\vec{x}, \vec{y}$ with limit zero, then $\lim_{n\to\infty} c_1 x_n + c_2 y_n = c_1 \lim_{n\to\infty} x_n + c_2 \lim_{n\to\infty} y_n = 0 + 0 = 0$. Then $c_1\vec{x} + c_2\vec{y}$ is in V. Proof complete by the subspace criterion.

(b) Let c_1, c_2 be scalars and $\vec{u}, \vec{v}$ in V. Then $T(c_1\vec{u} + c_2\vec{v})$

$$= \begin{pmatrix} 0 \\ 0 \\ c_1 u_0 + c_2 v_0 \\ \vdots \end{pmatrix} = c_1 \begin{pmatrix} 0 \\ 0 \\ u_0 \\ \vdots \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 0 \\ v_0 \\ \vdots \end{pmatrix} = c_1 T(\vec{u}) + c_2 T(\vec{v}).$$

(c) $\ker(T) = \{ \vec{x} : T(\vec{x}) = \vec{0} \} = \{ \vec{x} : \text{all } x_n = 0 \} = \{\vec{0}\}$

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3. (Orthogonality, Gram-Schmidt) Complete two.

(a) [50%] Give an algebraic proof, depending only on inner product space properties, of the Cauchy-Schwartz inequality $|\mathbf{u}\cdot\mathbf{v}| \le \|\mathbf{u}\|\|\mathbf{v}\|$ in $\mathcal{R}^n$.

(b) [50%] Find the orthogonal projection of $\begin{pmatrix}1\\0\\0\end{pmatrix}$ onto $V = \mathbf{span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\-1\end{pmatrix}, \begin{pmatrix}-1\\1\\-1\end{pmatrix}\right\}$.

(c) [50%] If you did (a) and (b), then stop with 100%. Otherwise continue.

Find the QR-factorization of $A = \begin{pmatrix}4 & 5 & 0\\0 & 0 & -2\\3 & -5 & 0\end{pmatrix}$.

(e) [50%] If you did two already, then stop with 100%. Otherwise continue.

An $n \times n$ matrix A is said to to be orthogonal provided $\|A\mathbf{x}\| = \|\mathbf{x}\|$ for all $\mathbf{x}$. Prove that the product of orthogonal matrices is orthogonal.

(f) [50%] If you did two already, then stop with 100%. Otherwise continue.

Find a non-invertible matrix A having two QR-factorizations.

(a) Expand $(u+tv)\cdot(u+tv) \ge 0$ to get $\|u\|^2 + t^2\|v\|^2 + 2t(u\cdot v) \ge 0$. assume $\|v\| > 0$. Let t = root of $2t\|v\|^2 + 2(u\cdot v) = 0$. Substitute it. Then $\|u\|^2 + \|v\|^2\left(\frac{u\cdot v}{|v|^2}\right)^2 - 2\frac{(u\cdot v)^2}{|v|^2} \ge 0$ implies $|u\cdot v| \le \|u\|\,\|v\|$

If $\|v\| = 0$, Then $v = 0$ and $|u\cdot v| = 0 = \|u\|\,\|v\|$.

(b) Let the vectors be labeled v_1, v_2, v_3. Then $v_1\cdot v_2 = 0$ implies Gram-Schmidt vectors $u_1 = \frac{v_1}{\sqrt{3}}$, $u_2 = \frac{v_2}{\sqrt{2}}$. Then $v_3^\perp = v_3 + \frac{1}{3}v_1 - 0v_2$ $= \frac{2}{3}\begin{pmatrix}-1\\2\\-1\end{pmatrix}$ and $u_3 = \frac{1}{\sqrt{6}}\begin{pmatrix}-1\\2\\-1\end{pmatrix}$.

Let $\vec{x} = \begin{pmatrix}1\\0\\0\end{pmatrix}$. Then

$$\mathrm{proj}_V(\vec{x}) = (x\cdot u_1)u_1 + (x\cdot u_2)u_2 + (x\cdot u_3)u_3$$
$$= \frac{u_1}{\sqrt{3}} + \frac{u_2}{\sqrt{2}} - \frac{u_3}{\sqrt{6}} = \begin{pmatrix}1\\0\\0\end{pmatrix}$$

What I expected: Independent vectors implies $V = \mathbb{R}^3$, so $\vec{v} = \begin{pmatrix}1\\0\\0\end{pmatrix} \in V$ and then $\mathrm{Proj}_V(\vec{v}) = \vec{v}$.

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3. (Orthogonality, Gram-Schmidt) Complete two.

(a) [50%] Give an algebraic proof, depending only on inner product space properties, of the Cauchy-Schwartz inequality $|\mathbf{u}\cdot\mathbf{v}| \le \|\mathbf{u}\|\|\mathbf{v}\|$ in $\mathcal{R}^n$.

(b) [50%] Find the orthogonal projection of $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ onto $V = \mathbf{span}\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}\right\}$.

(c) [50%] If you did (a) and (b), then stop with 100%. Otherwise continue.

Find the QR-factorization of $A = \begin{pmatrix} 4 & 5 & 0 \\ 0 & 0 & -2 \\ 3 & -5 & 0 \end{pmatrix}$.

(d) [50%] If you did two already, then stop with 100%. Otherwise continue.

An $n \times n$ matrix A is said to to be orthogonal provided $\|A\mathbf{x}\| = \|\mathbf{x}\|$ for all $\mathbf{x}$. Prove that the product of orthogonal matrices is orthogonal.

(e) [50%] If you did two already, then stop with 100%. Otherwise continue.

Find a non-invertible matrix A having two QR-factorizations.

(c) $Q = \begin{pmatrix} 4/5 & 3/5 & 0 \\ 0 & 0 & 1 \\ 3/5 & -4/5 & 0 \end{pmatrix}$ $R = \begin{pmatrix} 5 & a & b \\ 0 & 7 & c \\ 0 & 0 & 2 \end{pmatrix}$ $a = u_1\cdot v_2 = 1$, $b = u_1 \cdot v_3 = 0$, $c = u_2\cdot v_3 = 0$

(d) Let A, B be orthogonal. Then $\|AB\mathbf{x}\| = \|B\mathbf{x}\|$ because A is orthogonal. Then $\|AB\mathbf{x}\| = \|B\mathbf{x}\| = \|\mathbf{x}\|$ because B is orthogonal.

(e) There is no such matrix. If $A = QR$ and A has independent columns, Q orthogonal, and R upper triangular with positive diagonal entries, then Q, R are uniquely determined by A.

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4. **(Orthogonality and least squares)** Complete both.

(a) [50%] Assume $\mathbf{ker}(A) = \{\mathbf{0}\}$. Prove that the least squares normal equation for an inconsistent system $A\mathbf{x} = \mathbf{b}$ has a unique solution and display this solution.

(b) [50%] Prove the *near point theorem*: Given a vector $\mathbf{x}$ in $\mathcal{R}^3$ and a subspace V of $\mathcal{R}^3$, then $\mathbf{v} = \mathbf{proj}_V(\mathbf{x})$ is the nearest point in V to $\mathbf{x}$. This statement means that the minimum of $\|\mathbf{x} - \mathbf{v}\|$ is attained over all $\mathbf{v}$ in V at precisely the one point $\mathbf{v} = \mathbf{proj}_V(\mathbf{x})$.

(a) $\ker(A) = \{0\} \Rightarrow \ker(A^T A) = \{0\} \Rightarrow A^T A$ is invertible.

Then $A^T A x = A^T b$ (normal eq) has unique sol

$x = (A^T A)^{-1} A^T b$

(b) We have $x - \mathrm{proj}_V(x) \perp V$ and for any $v \in V$ also $\mathrm{proj}_V(x) - v \in V$. Therefore $x - \mathrm{proj}_V(x) \perp \mathrm{proj}_V(x) - v$.

By the Pythagorean theorem,

$$\|x - \mathrm{proj}_V(x)\|^2 + \|\mathrm{proj}_V(x) - v\|^2 = \|x - v\|^2$$

Then

$$\|x - \mathrm{proj}_V(x)\| < \|x - v\|$$

unless $\mathrm{proj}_V(x) - v = 0$. Therefore, $v = \mathrm{proj}_V(x) =$ minimum.

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5. (Determinants) Complete two.

(a) [50%] Given a 7×7 triangular matrix A, let B be obtained from A by a finite number of row swaps. Report all possible values of $\det(B)$, then prove your statement.

(b) [50%] Find A^{-1} by two methods: via the classical adjoint formula $A^{-1} = \mathbf{adj}(A)/\det(A)$, and the frame sequence method C to $\mathbf{rref}(C)$, applied to $C = \mathbf{aug}(A, I)$:

$$A = \begin{pmatrix} 4 & 2 & 0 \\ 0 & 0 & -2 \\ 3 & -2 & 0 \end{pmatrix}.$$

(c) [50%] If you did (a) and (b), then stop with 100%. Otherwise continue.
Let 4×4 matrices A and B be given and assume $B = E_4 E_3 E_2 E_1 A$. The elementary matrices E_1, E_2, E_3, E_4 represent combo(1,3,-15), swap(1,4), mult(2,-1/4), mult(3,-5), respectively. Find $\det(2B^2A)$, given $\det(A) = 5$.

(a) $B = E_1 \cdots E_k A$ where $E_1, \ldots, E_k$ are swaps. Then
$\det(B) = \det(E_1) \cdots \det(E_k) \det(A) = (-1)^k \det(A)$.
Report : $\det(B) = \pm \det(A)$.

(b)
$$\begin{pmatrix} 4 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 & 1 & 0 \\ 3 & -2 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 4 & 0 & 1 & 0 & -1 \\ 0 & 0 & -2 & 0 & 1 & 0 \\ 3 & -2 & 0 & 0 & 0 & 1 \end{pmatrix} \text{combo}$$

$$\begin{pmatrix} 1 & 4 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1/2 & 0 \\ 0 & 14 & 0 & 3 & 0 & -4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 4 & 0 & 1 & 0 & -1 \\ 0 & 14 & 0 & 3 & 0 & -4 \\ 0 & 0 & 1 & 0 & -1/2 & 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{7} & 0 & \frac{1}{7} \\ 0 & 1 & 0 & \frac{3}{14} & 0 & -2/7 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \end{array}\right)$$

$$\mathrm{adj}(A) = \begin{pmatrix} -4 & -6 & 0 \\ 0 & 0 & 14 \\ -4 & 8 & 0 \end{pmatrix}^T$$

$$\det(A) = 2 \begin{vmatrix} 4 & 2 \\ 3 & -2 \end{vmatrix} = -28$$

$$A^{-1} = \begin{pmatrix} -4 & 0 & -4 \\ -6 & 0 & 8 \\ 0 & 14 & 0 \end{pmatrix} \left(\frac{-1}{28}\right)$$

(c) $\det(B) = \det(E_4)\det(E_3)\det(E_2)\det(E_1)\det(A)$
$= (-5)(-1/4)(-1)(1)\det(A)$
$= -\frac{5}{4}\det(A)$
$= -\frac{25}{4}$

$\det(2B^2A) = \det(2I)\det(B)^2\det(A)$
$= 2^4\left(\frac{25}{4}\right)^2 5$
$= (25)^2 5$
$= 5^5$