Introduction to Linear Algebra 2270-2
Revised Sample, Midterm Exam 2 Spring 2007
Exam Date: 28 March

**Instructions.** This exam is designed for 50 minutes. Calculators, books, notes and computers are not allowed.

1. *(Matrices, determinants and independence)* Do two parts.

   (a) Prove that the pivot columns of $A$ form a basis for $\text{im}(A)$.

   (b) Suppose $A$ and $B$ are both $n \times m$ of rank $m$ and $\text{rref}(A) = \text{rref}(B)$. Prove or give a counterexample: the column spaces of $A$ and $B$ are identical.

Start your solution on this page. Please staple together any additional pages for this problem.
2. (Kernel and similarity) Do two parts.

(a) Illustrate the relation \( \text{rref}(A) = E_k \cdots E_2 E_1 A \) by a frame sequence and explicit elementary matrices for the matrix

\[
A = \begin{pmatrix}
0 & 1 & 2 \\
1 & 1 & 0 \\
2 & 2 & 0
\end{pmatrix}.
\]

(b) Prove or disprove: \( \ker(\text{rref}(BA)) = \ker(A) \), for all invertible matrices \( B \).
3. (Independence and bases) Do two parts.
   (a) Let $A$ be a $12 \times 15$ matrix. Suppose that, for any possible independent set $v_1, \ldots, v_k$, the set $Av_1, \ldots, Av_k$ is independent. Prove or give a counterexample: $\ker(A) = \{0\}$.
   (b) Let $V$ be the vector space of all polynomials $c_0 + c_1x + c_2x^2$ under function addition and scalar multiplication. Prove that $1 - x, 2x, (x - 1)^2$ form a basis of $V$.

Start your solution on this page. Please staple together any additional pages for this problem.
4. (Linear transformations) Do two parts.
   (a) Let $L$ be a line through the origin in $\mathbb{R}^3$ with unit direction $\mathbf{u}$. Let $T$ be a reflection through $L$. Define $T$ precisely. Display its representation matrix $A$, i.e., $T(x) = Ax$.
   (b) Let $T$ be a linear transformation from $\mathbb{R}^n$ into $\mathbb{R}^m$. Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be the columns of $I$ and let $A$ be the matrix whose columns are $T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)$. Prove that $T(x) = Ax$. 

Start your solution on this page. Please staple together any additional pages for this problem.
5. (Vector spaces)
   (a) Show that the set of all $4 \times 3$ matrices $A$ which have exactly one element equal to 1, and all other elements zero, form a basis for the vector space of all $4 \times 3$ matrices.

   (b) Let $S = \left\{ \begin{pmatrix} a & b \\ -a & 2b \end{pmatrix} : a, b \text{ real} \right\}$. Find a basis for $S$.

   (c) Let $V$ be the vector space of all functions defined on the real line, using the usual definitions of function addition and scalar multiplication. Let $S$ be the set of all polynomials of degree less than 5 (e.g., $x^4 \in V$ but $x^5 \notin V$) that have zero constant term. Prove that $S$ is a subspace of $V$. 

Start your solution on this page. Please staple together any additional pages for this problem.