Name ________________________ **2270-2**

Introduction to Linear Algebra 2270-2
Sample Midterm Exam 2 Spring 2007
Exam Date: 28 March

Instructions. This exam is designed for 50 minutes. Calculators, books, notes and computers are not allowed.

1. (Matrices, determinants and independence) Do two parts.

(a) Assume that $\det(EA) = \det(E)\det(A)$ holds for an elementary swap, multiply or combination matrix E and any square matrix A. Let $B = E_3E_2E_1C$ where $\det(C) = 4$ and E_1, E_2, E_3 represent `combo(1,2,-1), mult(3,-2), swap(1,3)` respectively. Find $\det(B^{-1})$.

(b) Prove that the pivot columns of A form a basis for $\mathbf{im}(A)$.

(c) Suppose A and B are both $n\times m$ of rank m and $\mathbf{rref}(A) = \mathbf{rref}(B)$. Prove or give a counterexample: the column spaces of A and B are identical.

(d) Let T be the linear transformation on $\mathcal{R}^3$ defined by mapping the columns of the identity respectively into three independent vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$. Define $\mathbf{u}_1 = \mathbf{v}_1 + 2\mathbf{v}_3$, $\mathbf{u}_2 = \mathbf{v}_1 + 3\mathbf{v}_2$, $\mathbf{u}_3 = \mathbf{v}_2 + 4\mathbf{v}_3$. Verify that $\mathcal{B} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a basis for $\mathcal{R}^3$ and report the $\mathcal{B}$-matrix of T (Otto Bretscher page 139).

Start your solution on this page. Please staple together any additional pages for this problem.

2. (Kernel and similarity) Do two parts.

(a) Illustrate the relation $\mathbf{rref}(A) = E_k \cdots E_2 E_1 A$ by a frame sequence and explicit elementary matrices for the matrix

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 2 & 2 & 0 \end{pmatrix}.$$

(b) Prove or disprove: $\mathbf{ker}(\mathbf{rref}(BA)) = \mathbf{ker}(A)$, for all invertible matrices B.

(c) Find a matrix A of size 3×3 that is not similar to a diagonal matrix. Verify assertions.

(d) Let

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Prove or disprove: A is similar to the upper triangular matrix T.

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3. (Independence and bases) Do two parts.

(a) Let A be a 12×15 matrix. Suppose that, for any possible independent set $\mathbf{v}_1, \ldots, \mathbf{v}_k$, the set $A\mathbf{v}_1, \ldots, A\mathbf{v}_k$ is independent. Prove or give a counterexample: $\mathbf{ker}(A) = \{\mathbf{0}\}$.

(b) Let V be the vector space of all polynomials $c_0 + c_1 x + c_2 x^2$ under function addition and scalar multiplication. Prove that $1 - x$, $2x$, $(x-1)^2$ form a basis of V.

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4. **(Linear transformations)** Do two parts.

(a) Let L be a line through the origin in $\mathcal{R}^3$ with unit direction $\mathbf{u}$. Let T be a reflection through L. Define T precisely. Display its representation matrix A, i.e., $T(\mathbf{x}) = A\mathbf{x}$.

(b) Let T be a linear transformation from $\mathcal{R}^n$ into $\mathcal{R}^m$. Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be the columns of I and let A be the matrix whose columns are $T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)$. Prove that $T(\mathbf{x}) = A\mathbf{x}$.

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5. (Vector spaces)

(a) Show that the set of all 4×3 matrices A which have exactly one element equal to 1, and all other elements zero, form a basis for the vector space of all 4×3 matrices.

(b) Let $S = \{\begin{pmatrix} a & b \\ -a & 2b \end{pmatrix} : a, b \ \text{ real}\}$. Find a basis for S.

(c) Let V be the vector space of all functions defined on the real line, using the usual definitions of function addition and scalar multiplication. Let S be the set of all polynomials of degree less than 5 (e.g., $x^4 \in V$ but $x^5 \notin V$) that have zero constant term. Prove that S is a subspace of V.

Start your solution on this page. Please staple together any additional pages for this problem.