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Introduction to Linear Algebra 2270-2
Midterm Exam 2 Spring 2007
Exam Date: 28 March

Instructions. This exam is designed for 50 minutes. Calculators, books, notes and computers are not allowed.

1. **(Matrices and independence)** Do two parts.
(a) It is known that each pivot column j of A satisfies the relation $E\,\mathbf{col}(A,j) = \mathbf{col}(I,j)$, for some invertible matrix E. Identify E from the theory of frame sequences and prove the relation.
(b) Suppose A and B are both $n \times n$ of rank n. Prove or give a counterexample: the column spaces of A and B are identical.

(a) Start with $\mathrm{rref}(A) = E_k \cdots E_1 A$, the frame sequence identity written in terms of elementary matrices. Define $E = E_k \cdots E_1$. Because elementary matrices are invertible, then E^{-1} exists. Pivot columns of A are the column numbers in $\mathrm{rref}(A)$ corresponding to leading ones. Such columns are columns of the identity. Then $\mathrm{rref}(A) = EA$ implies $\mathrm{col}(I,j) = E\,\mathrm{col}(A,j)$.

(b) Proof: The columns of A are independent, because row rank equals column rank. But n vectors independent in an n-dimension space form a basis, then $\mathrm{Colspace}(A) = \mathbb{R}^n$. The same remarks apply to B. Therefore,

$$\mathrm{Colspace}(A) = \mathbb{R}^n = \mathrm{colspace}(B).$$

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2. (Kernel and similarity) Do three parts.

(a) Illustrate the relation $\mathbf{rref}(A) = E_k \cdots E_2 E_1 A$ by a frame sequence and explicit elementary matrices for the matrix

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 2 \\ 2 & 2 & 0 \end{pmatrix}.$$

(b) Prove or disprove: $\mathbf{ker}(B) = \mathbf{ker}(A)$, for all frames A and B in any frame sequence.

(c) Prove or disprove: If $B = S^{-1}AS$ for an invertible square matrix S, then $\mathbf{ker}(A) = \mathbf{ker}(B)$.

(a) One such is $\mathrm{rref}(A) = E_5E_4E_3E_2E_1A$ where $E_1 = \mathrm{swap}(1,2)$, $E_2 = \mathrm{combo}(1,3,-2)$, $E_3 = \mathrm{mult}(3,-\frac{1}{2})$, $E_4 = \mathrm{combo}(2,3,-1)$, $E_5 = \mathrm{combo}(2,1,-2)$. There is not a unique answer to this problem.

(b) Swap, combo, mult do not create or destroy solutions, therefore all frames have the same solution set, that is, $\ker(A) = \ker(B)$ for all frames A, B.

(c) If $Bx = 0$, then $S^{-1}ASx = 0$ implies $ASx = 0$, which proves $Sx \in \ker(A)$. So $\ker(B) \subseteq \ker(AS) = \ker(SB)$. Not True.

Example: $\ker(A) = \mathrm{span}\left\{\begin{pmatrix}0\\1\end{pmatrix}\right\}$, $\ker(B) = \mathrm{span}\left\{\begin{pmatrix}-2\\1\end{pmatrix}\right\}$

$A = \begin{pmatrix}0 & 2\\0 & 3\end{pmatrix}$, $B = \begin{pmatrix}1 & 2\\1 & 2\end{pmatrix}$, $S = \begin{pmatrix}1 & 1\\1 & 2\end{pmatrix}$, $S^{-1} = \begin{pmatrix}2 & -1\\-1 & 1\end{pmatrix}$

$SB = \begin{pmatrix}1 & 1\\1 & 2\end{pmatrix}\begin{pmatrix}1 & 2\\1 & 2\end{pmatrix} = \begin{pmatrix}2 & 4\\3 & 6\end{pmatrix}$

$AS = \begin{pmatrix}0 & 2\\0 & 3\end{pmatrix}\begin{pmatrix}1 & 1\\1 & 2\end{pmatrix} = \begin{pmatrix}2 & 4\\3 & 6\end{pmatrix}$

Then $SB = AS$ or $B = S^{-1}AS$.

What's true? $\ker(B) = \ker(AS)$

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3. (Independence and bases) Do two parts.

(a) Let A be a 15×11 matrix. Prove or give a counterexample: $\dim(\mathbf{ker}(A)) + \dim(\mathbf{im}(A)) = 11$.

(b) Let V be the vector space of all polynomials $p(x) = c_0 + c_1 x + c_2 x^2$ under function addition and scalar multiplication. Let S be the subspace of V satisfying the relation $\int_{-1}^{1} p(x)dx = p(1)$. Find $\dim(S)$ and display a basis for S.

(a) $\dim(\mathrm{Im}(A)) =$ # pivot cols of A, because they are a basis for $\mathrm{Im}(A)$

$=$ # leading ones in rref(A)

$= \mathrm{rank}(A)$

$\dim(\ker(A)) =$ # free vars from rref(A)

$= 11 - \mathrm{rank}(A)$

$\therefore$ The two add to 11, as written. The proof is complete.

(b) $$\int_{-1}^{1} p\,dx = 2c_0 + c_1 \frac{x^2}{2}\Big|_{-1}^{1} + c_2 \frac{x^3}{3}\Big|_{-1}^{1}$$

$$= 2c_0 + \frac{2}{3}c_2$$

$$p(1) = c_0 + c_1 + c_2$$

The relation implies $c_0 + (-1)c_1 + (\frac{2}{3} - 1)c_2 = 0$

write this as $A\vec{c} = \vec{0}$ and solve for $\vec{c}$. Then c_1, c_2 are free vars t_1, t_2 and $\begin{cases} c_0 = t_1 + \frac{1}{3}t_2 \\ c_1 = t_1 \\ c_2 = t_2 \end{cases}$ is the gen sol.

Define the basis $\mathcal{B} = \{1 + x, \frac{1}{3} + x^2\}$. Then $\dim(S) = 2$.

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4. (Linear transformations) Do two parts.

(a) Let L be a line through the origin in $\mathcal{R}^4$ with unit direction $\mathbf{u}$. Let $T(\mathbf{x})$ be the projection of $\mathbf{x}$ onto L. Define T precisely. Display its representation matrix A, i.e., $T(\mathbf{x}) = A\mathbf{x}$.

(b) Let T be a linear transformation from $\mathcal{R}^n$ into $\mathcal{R}^m$. Let C be an invertible $n \times n$ matrix whose columns are $\mathbf{v}_1, \ldots, \mathbf{v}_n$. Let A be the matrix whose columns are $T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)$. Find a matrix B, written in terms of A and C, such that $T(\mathbf{x}) = B\mathbf{x}$. Your answer must provide a proof that $T(\mathbf{x}) = B\mathbf{x}$.

(a) $T(\vec{x}) = (\vec{x}\cdot\vec{u})\,\vec{u}$

$$A = \begin{pmatrix} u_1u_1 & \cdots & u_4u_1 \\ u_1u_2 & \cdots & u_4u_2 \\ u_1u_3 & \cdots & u_4u_3 \\ u_1u_4 & \cdots & u_4u_4 \end{pmatrix}$$

(b) Every $\vec{x} \in \mathbb{R}^n$ can be written as

$$\vec{x} = \sum_{i=1}^{n} a_i v_i = C\vec{a}$$

Then

$$T(\vec{x}) = \sum a_i T(v_i) = A\vec{a} = AC^{-1}C\vec{a} = AC^{-1}\vec{x}$$

So $\boxed{B = AC^{-1}}$

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5. (Vector spaces)

(a) Let V be the vector space of all 2×4 matrices A. Display a basis of V in which each basis element has exactly two nonzero entries. Include a proof that the displayed set is indeed a basis.

(b) Let $S = \left\{ \begin{pmatrix} a & b-c \\ -a & 2b-2c+a \end{pmatrix} : a, b, c \text{ real} \right\}$. Find a basis for S.

(c) Let V be the vector space of all functions defined on the real line, using the usual definitions of function addition and scalar multiplication. Let $V_1 = \mathbf{span}\{1, x, 1-x, e^x\}$ and let S be the subset of all functions $f(x)$ in V_1 such that $f(0) = f(1)$. Prove that S is a subspace of V.

Idea: Choose a basis $u_1, \ldots, u_8$. Then Let $v_1 = u_1 + u_2, \ldots, v_7 = u_1 + u_7, v_8 = -u_1 + u_2$

(a) $\begin{pmatrix} 1&1&0&0 \\ 0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&1&0 \\ 0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1&1 \\ 0&0&0&0 \end{pmatrix}, \ldots, \begin{pmatrix} 0&0&0&0 \\ 0&0&1&1 \end{pmatrix}$ makes 7 basis elements

For #8, choose $\begin{pmatrix} -1&1&0&0 \\ 0&0&0&0 \end{pmatrix}$. If $\sum_{i=1}^{8} c_i \vec{v}_i = \vec{0}$, Then

$$\begin{cases} c_1 - c_8 = 0 \\ c_1 + c_2 + c_8 = 0 \\ c_2 + c_3 = 0 \\ c_3 + c_4 = 0 \\ c_4 + c_5 = 0 \\ c_5 + c_6 = 0 \\ c_6 + c_7 = 0 \\ c_7 = 0 \end{cases} \Rightarrow \text{all } c_i = 0 \quad 1 \le i \le 8 \Rightarrow \text{elements } \vec{v}_1, \ldots, \vec{v}_8 \text{ are independent and form a basis.}$$

(b) candidates: $\partial_a, \partial_b, \partial_c \longrightarrow \begin{pmatrix} 1&0 \\ -1&1 \end{pmatrix}, \begin{pmatrix} 0&1 \\ 0&2 \end{pmatrix}, \begin{pmatrix} 0&-1 \\ 0&-2 \end{pmatrix}$.

Last one redundant. Choose $\mathcal{B} = \left\{ \begin{pmatrix} 1&0 \\ -1&1 \end{pmatrix}, \begin{pmatrix} 0&1 \\ 0&2 \end{pmatrix} \right\}$.

(c) V_1 has basis $1, x, e^x$. Define $T(c_1 + c_2 x + c_3 e^x) = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$ from V_1 onto $\mathbb{R}^3$. Then T is an isomorphism. The condition $f(0) = f(1)$ means $c_1 + c_3 = c_1 + c_2 + c_3 e$, or $c_2 + (e-1)c_3 = 0$. This is the kernel of $A = \begin{pmatrix} 0&1&e-1 \\ 0&0&0 \\ 0&0&0 \end{pmatrix}$, hence a subspace of $\mathbb{R}^3$. Because $T(S) = \ker(A)$, Then S is a subspace.

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