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Applied Linear Algebra 2270-2
Sample Midterm Exam 1
Wednesday, 14 Feb 2007

Instructions: This in-class exam is 50 minutes. No tables, notes, books or calculators allowed. The problems below contain more items than can be solved in 50 minutes. Some problems are too hard for an in-class exam, others are too easy. This is a sample only, and not an actual exam.

1. (Inverse of a matrix) Supply details for three of these:

- **a**. Define inverse for a matrix A.
- **b**. A matrix cannot have two inverses.
- **c**. Given $A^2 = 0$, find the inverse of $I + A$.
- **d**. If A and B are two matrices, not necessarily square, such that $AB = I$, then $B\mathbf{x} = \mathbf{0}$ has unique solution $\mathbf{x} = \mathbf{0}$.
- **e**. Give an example of a 3×3 matrix A and a frame sequence from $C = \mathbf{aug}(A, I)$ to $\mathbf{rref}(C)$ which produces a formula for A^{-1}.

Please staple this page to your solution. Write your initials on all pages.

Name. ______________________

2. **(Elementary Matrices)** Let A be a 3×3 invertible matrix. Let $\mathbf{rref}(A)$ be obtained from A by the following sequential row operations: (1) Swap rows 1 and 3; (2) Add -3 times row 2 to row 1; (3) Add 2 times row 3 to row 1; (4) Multiply row 2 by π.

- **a**. Write a matrix multiplication formula for A^{-1} in terms of explicit elementary matrices. (80%)
- **b**. Find A. (20%)

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Name. ________________________

3. (RREF method)
Let f, g and h denote constants and consider the system of equations

$$\begin{pmatrix} 1 & g-h & f \\ 0 & h & f \\ 1 & g & 2f \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -f \\ f \\ 0 \end{pmatrix}$$

a. Determine those values of f, g and h such that the system has a solution. (65%)

b. For each of the values in **a**, display the solution x, y, z to the system. (25%)

c. For each of the values in **a**, display the rank and nullity of the system. (10%)

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4. (Matrix algebra)
Do two of these:

a. Find all invertible 2×2 matrices A such that $A^{-1}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} A = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$.

b. Find all matrices B that commute with $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

c. Is $(A+B)(A-B) = A^2 - B^2$ true for square matrices A and B? Prove or give a counterexample.

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5. (Geometry and linear transformations)

Classify $T(\mathbf{x}) = A\mathbf{x}$ geometrically as scaling, projection onto line L, reflection in line L, pure rotation by angle θ, rotation composed with scaling, horizontal shear, vertical shear.

a. $A = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$

b. $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

c. $A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

d. $A = \begin{pmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{pmatrix}$

e. $A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$

f. $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

g. $A = \begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix}$

For additional problems of this type, and related questions, please study the first 20 exercises in 2.2.

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Extra Problems for Study

1x. (Inverse of a matrix) An $n \times n$ matrix A is said to have an inverse B if $AB = BA = I$, where I is the $n \times n$ identity matrix. Prove these facts:

1. If B_1 and B_2 are inverses of A, then $B_1 = B_2$.

2. The inverse of the identity I is I.

3. The zero matrix has no inverse.

4. In checking the inverse relation $AB = BA = I$, only one of $AB = I$ or $BA = I$ needs to be verified. You may apply a theorem from the textbook.

2x. (Elementary Matrices) Let A be a 3×3 matrix and $\vec{b}$ a vector in $\mathcal{R}^3$. Define $C = \mathbf{aug}(A, \vec{b})$. Let matrix F be obtained from C by the following: (a) Swap rows 2 and 3; (b) Add -1 times row 3 to row 1; (c) Swap rows 1 and 2; (d) Multiply row 2 by -5. Write a matrix multiplication formula for F in terms of C and explicit elementary matrices.

3x. (RREF method)
Let a and b denote constants and consider the system of equations

$$\begin{pmatrix} 1 & a+b & a \\ 0 & 0 & a \\ 1 & a+b & 2a \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ a \\ a \end{pmatrix}$$

(1) Determine those values of a and b such that the system has a solution.
(2) For each of the values in (1), solve the system.
(3) For each of the solutions in (2), check the answer.

4x. (rref)
(a) Determine a, b such that the system has a unique solution, infinitely many solutions, or no solution:

$$\begin{array}{rcrcrcl} x & + & 2y & + & z & = & b \\ 3x & + & 5y & + & 2z & = & 2b \\ 4x & + & ay & + & 3z & = & 2+b \end{array}$$

5x. (inverses)
Determine all values of x for which A^{-1} exists: $A = \begin{pmatrix} 1 & 2x-1 \\ 2 & -3 \end{pmatrix}$.

6x. (Frame sequence) Supply details for as many of these as time allows:

a. State the three possibilities for a linear system.

b. What are the three rules used to produce a frame sequence?

c. Give an example of a 3×3 matrix A and a frame sequence which ends in 4-10 steps with $\mathbf{rref}(A)$.

d. If A and B are two matrices, not necessarily square, such that $AB = I$, then $B\mathbf{x} = \mathbf{0}$ has unique solution $\mathbf{x} = \mathbf{0}$. Supply a proof or counterexample.

e. If A and B are two square matrices such that $AB = I$, then $A\mathbf{x} = \mathbf{0}$ has unique solution $\mathbf{x} = \mathbf{0}$. Supply a proof or counterexample.

f. If A and B are two non-square matrices such that $AB = I$, then $A\mathbf{x} = \mathbf{0}$ has unique solution $\mathbf{x} = \mathbf{0}$. Supply a proof or counterexample.

g. If A and B are two square 2×2 matrices such that $AB = I$, then $BA = I$. Supply a proof or counterexample.