Name. KEY

Applied Linear Algebra 2270-2
Midterm Exam 1
Wednesday, 14 Feb 2007

Instructions: This in-class exam is designed for 50 minutes. No tables, notes, books or calculators allowed.

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1. (Inverse of a matrix) Supply details for two of these:

a. If A and B are $n \times n$ invertible, then $(AB)^{-1} = B^{-1}A^{-1}$.

b. Give an example of two matrices A and B, not necessarily square, such that $AB = I$ but $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions.

c. Give an example of a 3×3 matrix A and a frame sequence starting at $C = \mathbf{aug}(A, I)$ which proves that A^{-1} does not exist.

(a) We show $C = AB$ and $D = B^{-1}A^{-1}$ satisfy $CD = DC = I$.

$$CD = ABB^{-1}A^{-1} = AIA^{-1} = AA^{-1} = I \qquad DC = B^{-1}A^{-1}AB = B^{-1}IB = B^{-1}B = I$$

(b) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. $A\begin{bmatrix} 0 \\ 0 \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ for all x.

(c) $\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$ $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Sequence has length 1.

A has an inverse $\Leftrightarrow$ $\mathrm{rref}(\mathrm{aug}(A, I)) = \mathrm{aug}(I, B)$ and $B = A^{-1}$. Since the last frame does not have I in left half, then A^{-1} does not exist.

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2. (Elementary Matrices) Let A be a 3×3 matrix. Let

$$\mathbf{rref}(A) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Assume $\mathbf{rref}(A)$ is obtained from A by the following sequential row operations: (1) Swap rows 1 and 2; (2) Add -3 times row 2 to row 3; (3) Add 2 times row 1 to row 2; (4) Multiply row 2 by 4.

a. Write a matrix multiplication formula for $\mathbf{rref}(A)$ in terms of explicit elementary matrices and the matrix A. (80%)

b. Find A. (20%)

(a) $$\mathrm{rref}(A) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} A$$

(b) $$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathrm{rref}(A)$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathrm{rref}(A)$$

$$= \begin{pmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ -6 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathrm{rref}(A)$$

$$= \begin{pmatrix} -2 & 1/4 & 0 \\ 1 & 0 & 0 \\ -6 & 3/4 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & -2 & 1/4 \\ 1 & 1 & 0 \\ -6 & -6 & 3/4 \end{pmatrix}$$

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3. (RREF method)

Part I. If a non-homogeneous system has a unique solution, then what is the rank and nullity of the corresponding homogeneous system? [20%]

Part II. Let a, b and c denote constants and consider the system of equations

$$\begin{pmatrix} 1 & b-c & a \\ 1 & c & -a \\ 2 & b & a \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -a \\ a \\ 0 \end{pmatrix}$$

a. Determine those values of a, b and c such that the system has a unique solution. (40%) $2c-b \neq 0$, $a \neq 0$

b. Determine those values of a, b and c such that the system has no solution. (20%)
$2c-b=0$, $a\neq 0$

c. Determine those values of a, b and c such that the system has infinitely many solutions. (20%) $a=0$ [value of $2c-b$ can be anything]

To save time, don't attempt to solve the equations or to produce a complete frame sequence.

part I rank = # vars, nullity = 0

part II parts a, b, c listed below.

$$\left(\begin{array}{ccc|c} 1 & b-c & a & -a \\ 1 & c & -a & a \\ 2 & b & a & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & b-c & a & -a \\ 0 & 2c-b & -2a & 2a \\ 0 & -b+2c & -a & 2a \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & b-c & a & -a \\ 0 & 2c-b & -2a & 2a \\ 0 & 0 & a & 0 \end{array}\right)$$

(a) $2c-b\neq 0$ and $a\neq 0 \Rightarrow$ 3 lead vars $\Rightarrow$ unique sol

case $2c-b=0$

$$\left(\begin{array}{ccc|c} 1 & c & a & -a \\ 0 & 0 & -2a & 2a \\ 0 & 0 & a & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & c & a & -a \\ 0 & 0 & a & -a \\ 0 & 0 & a & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & c & a & -a \\ 0 & 0 & a & -a \\ 0 & 0 & 0 & a \end{array}\right)$$

(b) No solution for $a\neq 0$ and $2c-b=0$ due to signal eq "$0=a$"

(c) Sequence 1 has a row of zeros for $a=0$
Sequence 2 has a row of zeros for $a=0$
In both cases, ∞-many sols.

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4. (Matrix algebra)
Do two of these:

a. Find all 2×2 matrices A such that $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A = A \begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}$.

b. Let A be a 3×2 matrix and B a 2×3 matrix. Explain using matrix algebra and the three possibilities why the 3×3 matrix $C = AB$ cannot be invertible.

c. Prove for 2×2 matrices A, B, C that $A(B + C) = AB + AC$. Please use only the definition of matrix addition, scalar multiply and matrix multiply.

(a) $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}$

$$\begin{pmatrix} c & d \\ a & b \end{pmatrix} = \begin{pmatrix} 3a & a+b \\ 3c & c+d \end{pmatrix}$$

$$\begin{array}{cccc} a & b & c & d \end{array}$$

$$\begin{pmatrix} 3 & 0 & -1 & 0 \\ 1 & 0 & -3 & 0 \\ 1 & 1 & 0 & -1 \\ 0 & -1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -3 & 0 \\ 3 & 0 & -1 & 0 \\ 1 & 1 & 0 & -1 \\ 0 & -1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -3 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 1 & 3 & -1 \\ 0 & -1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 4 & 0 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$a = 0$
$b = t_1$
$c = 0$
$d = t_1$

$$\boxed{A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} t_1 \text{ for all } t_1}$$

(b) suppose C is invertible. Because $\text{rank}(B) \le 2$, There is a nonzero $\vec{x}$ s.t. $B\vec{x} = \vec{0}$. Then

$$A B \vec{x} = C\vec{x}$$
$$A\vec{0} = C\vec{x}$$
$$\vec{0} = C\vec{x}$$
$$C^{-1}\vec{0} = \vec{x} \qquad \text{multiply by } C^{-1} \text{ both sides}$$
$$\vec{0} = \vec{x} \qquad \text{a contradiction.}$$

(c) Idea of proof: Let $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$, similar for B, C. Then

$$A(B+C) = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}\begin{pmatrix} b_1+c_1 & b_2+c_2 \\ b_3+c_3 & b_4+c_4 \end{pmatrix} = \begin{pmatrix} a_1b_1 + a_1c_1 + a_2b_3 + a_2c_3 & \vdots \\ \cdot & \cdot \end{pmatrix}$$

Expand $AB + AC$ and get The same matrix.

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5. (Geometry and linear transformations)

Part I. Classify $T(\mathbf{x}) = A\mathbf{x}$ geometrically as scaling, projection onto line L, reflection in line L, pure rotation by angle θ, rotation composed with scaling, horizontal shear, vertical shear. Define L and θ where applicable. [60%]

a. $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ for $\theta = 90^\circ$; $A\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$ c.c. $\theta = 90^\circ$

b. $A = \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ Horiz. Shear

c. $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \sqrt{2}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, $\theta = 45^\circ$ rotate cc 45°, scale $\sqrt{2}$

d. $A = \frac{1}{2}\begin{pmatrix} \sqrt{3} & 1 \\ 1 & -\sqrt{3} \end{pmatrix} = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}$, $a^2+b^2=1$ orthonormal cols Reflection in $x-(2+\sqrt{3})y=0$

Part II. Give details. [40%]

e. Define vertical and horizontal shear. $T(\vec{x}) = A\vec{x}$

$A = \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$ $A = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$

f. Display the matrix A of a projection onto a line L in $\mathcal{R}^3$.

$T(\vec{x}) = (\vec{x}\cdot\vec{u})\,\vec{u} = A\vec{x} \quad \Rightarrow \quad A = \mathrm{aug}(u_1\vec{u}, u_2\vec{u}, u_3\vec{u})$

g. Define reflection in a line L in $\mathcal{R}^2$.

$T(\vec{x}) = 2(x\cdot\vec{u})\,\vec{u} - \vec{x}$.

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