# Introduction to Linear Algebra 2270-2
# Final Exam 4:30pm 1 May 2007

**Instructions**. The time allowed is 120 minutes. The examination consists of six problems, one for each of chapters 3, 4, 5, 6, 7, 8, each problem with multiple parts. A chapter represents 20 minutes on the final exam. Each problem represents several textbook problems numbered (a), (b), (c), $\cdots$. Please solve enough parts to make 100% on each chapter. Choose the problems to be graded by check-mark $\boxed{\text{X}}$; the credits should add to 100.

Calculators, books, notes and computers are not allowed.

Answer checks are not expected or required. First drafts are expected, not complete presentations.

Please submit **exactly six** separately stapled packages of problems.

**Keep this page for your records.**

Name KEY                                                2270-2 Final Exam, 4:30pm, 1 May 2007

**Ch3. (Subspaces of $\mathcal{R}^n$ and Their Dimensions)**

☐ [30%] Ch3(a): Let $A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & -1 \end{bmatrix}$. Find bases for the image and kernel of $A$.

☐ [40%] Ch3(b): Let $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ the columns of the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. Define $T(\mathbf{x}) = A\mathbf{x}$. Find the matrix of $T$ relative to the basis $\mathbf{v}_1 + \mathbf{v}_2$, $\mathbf{v}_2 + \mathbf{v}_3$, $\mathbf{v}_3 + \mathbf{v}_1$.

☐ [30%] Ch3(d): Let $V$ be the vector space of all functions $f(x)$ defined on $0 \le x \le 1$. Let $S$ be the subset of $V$ defined by $f(1) = f(0) + \int_0^1 xf(x)dx$, $f(0.5) = 0$. Prove that $S$ is a subspace of $V$.

☐ [40% or 30%] Ch3(d): Let $V$ be the vector space of all data packages $\mathbf{v} = \begin{pmatrix} f \\ x_0 \\ y_0 \end{pmatrix}$, where $f$ is a continuous function defined on $0 \le x \le 1$ and $x_0$, $y_0$ are real values. Define $\boxed{+}$ and $\boxed{\cdot}$ componentwise. Let $S$ be the subset of $V$ defined by $f(0) = f(1)$, $2x_0 + y_0 = 0$. Prove that $S$ is a subspace of $V$.

Ch3(a) $\ker(A) = \text{span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \\ 1 \end{pmatrix} \right\}$. $\text{Im}(A) = \text{span}\{ \text{col}(A,1), \text{col}(A,2) \}$.

Ch3(b) $B = S^{-1}AS$, $S = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix}$, $S^{-1} = \frac{1}{4}\begin{pmatrix} -1 & -1 & 3 \\ -1 & 3 & -1 \\ 3 & -1 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

Ch3(c) $\vec{0}$ is $f(x) \equiv 0$, which satisfies both conditions $\Rightarrow$ $\vec{0} \in S$.
Let $\vec{v}_1, \vec{v}_2 \in S$, represented by $f_1, f_2$. Let $c_1, c_2$ = scalars. Define $f = c_1 f_1 + c_2 f_2$, which is the equation for $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2$. Then

$$f(1) = c_1 f_1(1) + c_2 f_2(1) = c_1\left(f_1(0) + \int_0^1 xf_1(x)dx\right) + c_2\left(f_2(0) + \int_0^1 xf_2(x)dx\right)$$
$$= f(0) + \int_0^1 x(c_1f_1 + c_2f_2)(x)\,dx = f(0) + \int_0^1 xf(x)dx$$
$$f(0.5) = c_1 f_1(0.5) + c_2 f_2(0.5) = 0 + 0 = 0$$

Therefore, $\vec{v} \in S$. The proof is complete, by the subspace criterion.

Ch3(d) $\vec{0} = \begin{pmatrix} \vec{0} \\ 0 \\ 0 \end{pmatrix}$ satisfies the equations $\Rightarrow$ $\vec{0}$ is in $S$.
Let $c_1, c_2$ = scalars and $\vec{v}_1, \vec{v}_2$ in $S$. Then $\vec{v}_1 = \begin{pmatrix} f_1 \\ x_1 \\ y_1 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} f_2 \\ x_2 \\ y_2 \end{pmatrix}$
Let $\vec{v} = \vec{v}_1 + \vec{v}_2 = \begin{pmatrix} f \\ x_0 \\ y_0 \end{pmatrix}$. Then

$$f(0) = c_1 f_1(0) + c_2 f_2(0) = c_1 f_1(1) + c_2 f_2(1) = f(1)$$

and

$$2x_0 + y_0 = 2(c_1 x_1 + c_2 x_2) + (c_1 y_1 + c_2 y_2)$$
$$= c_1(2x_1 + y_1) + c_2(2x_2 + y_2)$$
$$= 0 + 0 = 0$$

Therefore, $S$ is a subspace by the subspace criterion

Please start your solutions on this page. Staple on additional pages.

Name KEY

2270-2 Final Exam, 4:30pm, 1 May 2007

**Ch4. (Linear Spaces)**

☐ [30%] Ch4(a): Let $\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

Let $W$ be the linear space of all $3 \times 3$ matrices. Let $V$ be the set of all $3 \times 3$ matrices $A$ such that $\mathbf{x}$ belongs to the image of $A$. Prove or disprove : $V$ is a subspace of $W$.

☐ [40%] Ch4(b): Let $V$ be the linear space of all functions $f(x) = c_0 + c_1 x + c_2 x^2$. Define $T(f) = c_2 x^2$ from $V$ to $V$. Find the image, kernel, rank and nullity of $T$.

☐ [30%] Ch4(c): Let $V$ be the linear space of all real $4 \times 4$ matrices $M$. Let $T$ be defined on $V$ by $T(\mathbf{M}) = \mathbf{N}$ where $\mathbf{N} = \mathbf{M}$ except for the last row, which is all zeros. Find the image and kernel of $T$.

☐ [40%] Ch4(d): Let $A = \begin{pmatrix} 1 & 3 \\ 0 & -1 \end{pmatrix}$, $S = \begin{pmatrix} 1 & -3 \\ 0 & 2 \end{pmatrix}$ and $D = \mathbf{diag}(1, -1)$. Then $AS = SD$. Define $V$ to be the linear space of all $2 \times 2$ matrices $R$ satisfying $AR = RD$. Find a basis for $V$.

Ch4 (a) Zero is not in V. Not a subspace.

Ch4 (b) $\ker(T) = \text{span}\{1, x\}$. $\text{Im}(T) = \text{span}\{x^2\}$. Nullity $= 2$, rank $= 1$

Ch4 (c) $\ker(T) = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a & b & c & d \end{pmatrix} \right\}$ $\text{Im}(T) = \left\{ \begin{pmatrix} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}$ each $\bullet$ = variable. dimension $= 12$

Ch4 (d) Let $R = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $AR - RD = 0 \iff \begin{pmatrix} 3c & 2b+3d \\ -2c & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Then $R = \begin{pmatrix} a & -\frac{3}{2}d \\ 0 & d \end{pmatrix}$. Basis of $V = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -3 \\ 0 & 2 \end{pmatrix} \right\}$.

Please start your solutions on this page. Staple on additional pages.

Name KEY

2270-2 Final Exam, 4:30pm, 1 May 2007

## Ch5. (Orthogonality and Least Squares)

☐ [30%] Ch5(a): Find the orthogonal projection of $\mathbf{v}$ onto $V = \mathbf{span}(\mathbf{v}_1, \mathbf{v}_2)$, given

$$\mathbf{v} = \begin{pmatrix} 9 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v}_1 = \begin{pmatrix} 0 \\ 4 \\ -1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 4 \\ 1 \\ 0 \end{pmatrix}.$$

Expected method: $v \perp \mathrm{span}(v_1, v_2) \Rightarrow \mathrm{proj} = \vec{0}$

☐ [10%] Ch5(b): Let A be $4 \times 5$. Prove or give a counterexample: $\dim\left(\mathbf{im}(A)^{\perp}\right) = \dim\left(\mathbf{ker}\left(A^T\right)\right)$.

☐ [10%] Ch5(c): Let A be $n \times m$. Prove or give a counterexample: $\mathbf{ker}(A) = \mathbf{ker}\left(AA^T\right)$.

☐ [30%] Ch5(d): Consider the linear space $V$ of polynomials $f(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3$ on $0 \le t \le 1$ with inner product

$$< f, g >= \int_0^1 f(t)g(t)dt.$$

Find a basis for the subspace $S$ of all $f$ in $V$ orthogonal to both $t$ and $1+t$.

☐ [30%] Ch5(e): Find the Gram-Schmidt orthonormal vectors $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ for the following independent set:

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}.$$

☐ [30%] Ch5(f): Find the $QR$-factorization of $A = \begin{pmatrix} 4 & 10 & 0 \\ 0 & 0 & -1 \\ 3 & -10 & 0 \end{pmatrix}$.

☐ [30%] Ch5(g): Derive the normal equation in the theory of least squares.

Ch5(a) $\mathrm{proj}_V(\vec{v}) = \vec{0}$ because $u_1 = v_1/\|v_1\|$, $u_2 = v_2/\|v_2\|$ and $u_1 \cdot v = u_2 \cdot v = 0$.

Ch5(b) True, because $\mathrm{im}(A)^{\perp} = \ker(A^T)$ is a Theorem

Ch5(c) False, $AA^T$ does not make sense for $A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Ch5(d) $S = \{ f : < t, f > = < 1+t, f > = 0 \} = \{ f : < 1, f > = < t, f > = 0 \}$

$f_1 = -5 + 6t + 6t^2$, $f_2 = 2 - 9t + 10t^3$ $\quad S = \mathrm{span}\{f_1, f_2\}$.

Ch5(e) $\vec{w}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, $\vec{w}_2 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ -1 \\ 0 \\ 2 \end{pmatrix}$, $\vec{w}_3 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ -1 \\ 0 \\ -1 \end{pmatrix}$

Ch5(f) $Q = \frac{1}{5}\begin{pmatrix} 4 & 3 & 0 \\ 0 & 0 & -1 \\ 3 & -4 & 0 \end{pmatrix}$, $R = \begin{pmatrix} 5 & 2 & 0 \\ 0 & 14 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Ch5(g) See Bretscher, Fact 5.4.5, page 222.

Please start your solutions on this page. Staple on additional pages.

Name KEY

2270-2 Final Exam, 4:30pm, 1 May 2007

**Ch6. (Determinants)**

☐ [50%] Ch6(a): Let $B$ be the invertible matrix given below, where $\boxed{?}$ means the value of the entry does not affect the answer to this problem. The second matrix $C$ is the adjugate (or adjoint) of $B$. Find the value of $\det(2B^{-1}(B^T)^{-2})$.

$$B = \begin{pmatrix} ? & ? & ? & 0 \\ 0 & -1 & 2 & 0 \\ 1 & 1 & 0 & 0 \\ ? & ? & ? & 3 \end{pmatrix}, \quad C = \begin{pmatrix} 6 & 6 & 12 & 0 \\ -6 & -6 & 6 & 0 \\ -3 & 6 & 3 & 0 \\ 2 & 2 & 4 & -6 \end{pmatrix}$$

☐ [25%] Ch6(b): Assume $A = \mathbf{aug}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ is $3 \times 3$ and $B = \mathbf{aug}(\mathbf{v}_1 + \mathbf{v}_3, \mathbf{v}_3 - \mathbf{v}_2, 2\mathbf{v}_2 - \mathbf{v}_3)$. Suppose $\det(A + B) + (\det(A))^2 = 0$. Find all possible values of $\det(A)$.

Typo

☐ [25%] Ch6(c): Assume given $3 \times 3$ matrices $A$, $B$. Suppose $E_5E_4B = E_3E_2E_1A$ and $E_1$, $E_2$, $E_3$, $E_4$, $E_5$ are elementary matrices representing respectively a combination, a multiply by 3, a swap and a multiply by 7. Assume $\det(A) = 5$. Find $\det(5A^2B)$. combination,

☐ [25%] Ch6(d): Find the area of the parallelogram formed by $\mathbf{v}_1$, $\mathbf{v}_2$, given

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}.$$

☐ [25%] Ch6(e): Evaluate $\det(A)$ by any hybrid method.

$$A = \begin{pmatrix} 1 & 1 & 2 & 0 \\ -1 & -1 & 1 & 0 \\ -1 & 2 & 1 & 0 \\ 1 & 1 & 2 & -3 \end{pmatrix}$$

ch6(a) $BC = (18)I \Rightarrow \det(2B^{-1}(B^T)^{-2}) = \det(2I)\det(B)^{-1}\det(B^T)^{-2} = \frac{2^4}{18^3}$

ch6(b) $\det(A+B) = (-1)(2)(2)\det(A) \Rightarrow \det A = 0$ or $4$

ch6(c) $(7)(-1)\det B = (3)(1)(1)\det(A)$, $\det(5A^2B) = \det(5I)\det(A)^2\det B$
$= -\frac{3}{7}(5^6)$

ch6(d) $\|v_1\|^2 = 3$, $\|v_2\|^2 = 5$, area $= \sqrt{\|v_1\|^2\|v_2\|^2 - (v_1 \cdot v_2)^2} = \sqrt{15-1} = \sqrt{14}$

ch6(e) combo + cofactor, ans = 27

ch6(d) alternate: use area $= \sqrt{|\det(A^TA)|}$

Please start your solutions on this page. Staple on additional pages.

Name KEY 2270-2 Final Exam, 4:30pm, 1 May 2007

## Ch7. (Eigenvalues and Eigenvectors)

☐ [30%] Ch7(a): Find the eigenvalues of the matrix $A = \begin{pmatrix} 4 & -2 & 1 & 12 \\ 2 & 4 & -3 & 15 \\ 0 & 0 & 3 & 7 \\ 0 & 0 & -1 & -5 \end{pmatrix}$. To save time, **do not** find eigenvectors!

$\det(A-\lambda I) = ((4-\lambda)^2+4)(\lambda^2+2\lambda-8)$ $\boxed{\begin{matrix} 2, -4 \\ 4 \pm 2i \end{matrix}}$

☐ [30%] Ch7(b): Given $A = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix}$, assume there exists an invertible matrix $P$ and a diagonal matrix $D$ such that $AP = PD$. Circle all possible columns of $P$ from the list below.

$$\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}.$$

Test $A\vec{v} = \lambda \vec{v}$

☐ [40%] Ch7(c): Consider the $3 \times 3$ matrix

$$A = \begin{pmatrix} 4 & 2 & -2 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix}.$$

Already computed are eigenpairs

$$\left(2, \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}\right), \quad \left(4, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right).$$

(1) [25%] Find the remaining eigenpairs of $A$.
(2) [5%] Display an invertible matrix $P$ and a diagonal matrix $D$ such that $AP = PD$.
(3) [10%] Display explicitly Fourier's model for $A$.

☐ [40%] Ch7(d): Consider a discrete dynamical system $\mathbf{x}(n+1) = A\mathbf{x}(n)$. Given $A$ and $\mathbf{x}(0)$ below, find exact formulas for the vectors $\mathbf{x}(n)$ and $\lim_{n\to\infty} \mathbf{x}(n)$.

$$A = \frac{1}{10}\begin{pmatrix} 7 & 1 \\ -2 & 10 \end{pmatrix}, \quad \mathbf{x}(0) = \begin{pmatrix} 40 \\ 50 \end{pmatrix}.$$

Ch7(c) (1) $\left(4, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right)$, (2) $P = \begin{pmatrix} 2 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$,

(3) $A\left(c_1\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_3\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right) = 2c_1\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + 4c_2\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 4c_3\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

Ch7(d) $\vec{x}(n) = (PDP^{-1})^n \vec{x}(0) = PD^nP^{-1}\vec{x}(0)$, $P = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 9 & 0 \\ 0 & 8 \end{pmatrix} \cdot \frac{1}{10}$

$\lim_{n\to\infty} \vec{x}(n) = P \lim_{n\to\infty} D^n P^{-1}\vec{x}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Please start your solutions on this page. Staple on additional pages.

Name KEY

2270-2 Final Exam, 4:30pm, 1 May 2007

**Ch8. (Symmetric Matrices and Quadratic Forms)**

☐ [50%] Ch8(a): Find an orthonormal matrix $Q$ such that $Q^{-1}AQ$ is diagonal:

$$A = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}.$$

☐ [50%] Ch8(b): Find the ellipse semi-axis lengths $a$, $b$ and the unit semi-axis directions $\mathbf{v}_1$, $\mathbf{v}_2$ for the equation $3x^2 - 4xy + 6y^2 = 1$.

☐ [50%] Ch8(c): Find a singular value decomposition $A = U\Sigma V^T$ for

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Hint: $U$ is $3 \times 3$, $V$ is $2 \times 2$, and $\Sigma$ is $3 \times 2$.

☐ [50%] Ch8(d): Let $A = U\Sigma V^T$ be a singular value decomposition for an $n \times m$ matrix $A$ and let $\sigma_1, \ldots, \sigma_r$ be the positive singular values of $A$. Prove the formula

$$A = \sum_{j=1}^{r} \sigma_j \mathbf{u}_j \mathbf{v}_j^T,$$

where $U = \mathbf{aug}(\mathbf{u}_1, \ldots, \mathbf{u}_n)$ and $V = \mathbf{aug}(\mathbf{v}_1, \ldots, \mathbf{v}_m)$.

Ch8(a) $Q = \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix}$ $\lambda = 0, -1, -2$

No Gram-Schmidt required

Ch8(b) $A = \begin{pmatrix} 3 & -2 \\ -2 & 6 \end{pmatrix}$ Eigenpairs $\left(2, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right), \left(7, \begin{pmatrix} -1 \\ 2 \end{pmatrix}\right)$, $a = \frac{1}{\sqrt{2}}$, $b = \frac{1}{\sqrt{7}}$

$\vec{v}_1, \vec{v}_2$ directions given by the eigenvectors.

Ch8(c) $\Sigma = \begin{pmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$, $V = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, $U = \begin{pmatrix} 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \end{pmatrix}$

$A^TA = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ has eigenvalues 1, 3.

by Gram-Schmidt after adding $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

Ch8(d) See Bretscher 8.3-29 for a solution.

This appears in the sol manual.

Expected method: prove $Av = \sum \sigma_j u_j v_j^T v$ for all vectors $v$.

Details: $Av = U\Sigma V^T v = U\Sigma \begin{pmatrix} v_1^T v \\ \vdots \\ v_n^T v \end{pmatrix} = U \begin{pmatrix} \sigma_1 v_1^T v \\ \vdots \\ \sigma_n v_n^T v \end{pmatrix} = \sum_{j=1}^{n} (\sigma_j \cdot v_j^T \cdot v) u_j$

$= \sum_{j=1}^{r} (\sigma_j \cdot v_j^T v) u_j$ because $\sigma_{r+1} = \cdots = \sigma_n = 0$.

Please start your solutions on this page. Staple on additional pages.