

**Math 2250 Maple Project 5: Linear Algebra**  
**January 2007**

**Due date:** See the internet due dates. Maple lab 5 has problems L5.1, L5.2, L5.3.

**References:** Code in `maple` appears in 2250mapleL5-S2007.txt at URL <http://www.math.utah.edu/~gustafso/>. This document: 2250mapleL5-S2007.pdf.

**Problem L5.1. (Matrix Algebra)**

Define  $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ ,  $B = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$ ,  $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$  and  $\mathbf{w} = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$ . Create a worksheet in `maple` which states this problem in text, then defines the four objects. The worksheet should contain text, `maple` code and displays. Continue with this worksheet to answer (1)–(7) below. Submit problem L5.1 as a worksheet printed on 8.5 by 11 inch paper. See Example 1 for `maple` commands.

- (1) Compute  $AB$  and  $BA$ . Are they the same?
- (2) Compute  $A + B$  and  $B + A$ . Are they the same?
- (3) Let  $C = A + B$ . Compare  $C^2$  to  $A^2 + 2AB + B^2$ . Explain why they are different.
- (4) Compute transposes  $C_1 = (AB)^T$ ,  $C_2 = A^T$  and  $C_3 = B^T$ . Find an equation for  $C_1$  in terms of  $C_2$  and  $C_3$ . Verify the equation.
- (5) Solve for  $\mathbf{X}$  in  $B\mathbf{X} = \mathbf{v}$  by three different methods.
- (6) Solve  $A\mathbf{Y} = \mathbf{v}$  for  $\mathbf{Y}$ . Do an answer check.
- (7) Solve  $A\mathbf{Z} = \mathbf{w}$ . Explain your answer using rref theory.

**Problem L5.2. (Row space)**

Let  $A = \begin{pmatrix} 1 & 1 & 1 & 2 & 6 \\ 2 & 3 & -2 & 1 & -3 \\ 0 & 1 & -4 & -3 & -15 \\ 1 & 2 & -3 & -1 & -9 \end{pmatrix}$ . Find two different bases for the row space of  $A$ , using the following three methods.

1. The method of Example 2, below.
2. The `maple` command `rowSPACE(A)`.
3. The `rref`-method: select rows from `rref(A)`.

Two of the methods produce exactly the same basis. Verify that the two bases  $\mathcal{B}_1 = \{\mathbf{v}_1, \mathbf{v}_2\}$  and  $\mathcal{B}_2 = \{\mathbf{w}_1, \mathbf{w}_2\}$  are **equivalent**. This means that each vector in  $\mathcal{B}_1$  is a linear combination of the vectors in  $\mathcal{B}_2$ , and conversely, that each vector in  $\mathcal{B}_2$  is a linear combination of the vectors in  $\mathcal{B}_1$ .

**Problem L5.3. (Matrix Equations)**

Let  $A = \begin{pmatrix} 8 & 10 & 3 \\ -3 & -5 & -3 \\ -4 & -4 & 1 \end{pmatrix}$ ,  $T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$ . Let  $P$  denote a  $3 \times 3$  matrix. Assume the following result:

**Lemma 1.** The equality  $AP = PT$  holds if and only if the columns  $\mathbf{v}_1$ ,  $\mathbf{v}_2$ ,  $\mathbf{v}_3$  of  $P$  satisfy  $A\mathbf{v}_1 = \mathbf{v}_1$ ,  $A\mathbf{v}_2 = -2\mathbf{v}_2$ ,  $A\mathbf{v}_3 = 5\mathbf{v}_3$ . [proved after Example 4]

- (a) Determine three specific columns for  $P$  such that  $\det(P) \neq 0$  and  $AP = PT$ . Infinitely many answers are possible. See Example 4 for the `maple` method that determines a column of  $P$ .
- (b) After reporting the three columns, check the answer by computing  $AP - PT$  (it should be zero) and  $\det(P)$  (it should be nonzero).

**Example 1.** Let  $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 0 & -1 \end{pmatrix}$  and  $\mathbf{b} = \begin{pmatrix} 9 \\ 8 \\ 3 \end{pmatrix}$ . Create a **maple** work sheet. Define and display matrix  $A$  and vector  $\mathbf{b}$ . Then compute

- (1) The inverse of  $A$ .
- (2) The augmented matrix  $C = \mathbf{aug}(A, \mathbf{b})$ .
- (3) The reduced row echelon form  $R = \mathbf{rref}(C)$ .
- (4) The column  $\mathbf{X}$  of  $R$  which solves  $A\mathbf{X} = \mathbf{b}$ .
- (5) The matrix  $A^3$ .
- (6) The transpose of  $A$ .
- (7) The matrix  $A - 3A^2$ .
- (8) The solution  $\mathbf{X}$  of  $A\mathbf{X} = \mathbf{b}$  by two methods different than (4).

**Solution:** A lab instructor can help you to create a blank work sheet in **maple**, enter code and print the work sheet. The code to be entered appears below. To get help, enter ?linalg into a worksheet, then select commands that match ones below.

```
with(linalg):
A:=matrix([ [1,2,3], [2,-1,1], [3,0,-1] ]);
b:=vector([9,8,3]);
print("(1)"); inverse(A);
print("(2)"); C:=augment(A,b);
print("(3)"); R:=rref(C);
print("(4)"); X:=col(R,4);
print("(5)"); evalm(A^3);
print("(6)"); transpose(A);
print("(7)"); evalm(A-3*(A^2));
print("(8)"); X:=linsolve(A,b); X:=evalm(inverse(A) &* b);
```

**Example 2.** Let  $A = \begin{pmatrix} 1 & 1 & 1 & 2 & 6 \\ 2 & 3 & -2 & 1 & -3 \\ 3 & 5 & -5 & 1 & -8 \\ 4 & 3 & 8 & 2 & 3 \end{pmatrix}$ .

- (1) Find a basis for the column space of  $A$ .
- (2) Find a basis for the row space of  $A$ .
- (3) Find a basis for the nullspace of  $A$ .
- (4) Find  $\mathbf{rank}(A)$  and  $\mathbf{nullity}(A)$ .
- (5) Find the dimensions of the nullspace, row space and column space of  $A$ .

**Solution:** The theory applied: *The columns of  $B$  corresponding to the leading ones in  $\mathbf{rref}(B)$  are independent and form a basis for the column space of  $B$ .* These columns are called the **pivot columns** of  $B$ . Results for the row space can be obtained by applying the above theory to the transpose of the matrix.

The **maple** code which applies is

```
with(linalg):
A:=matrix([ [ 1, 1, 1, 2, 6],
[ 2, 3,-2, 1,-3],
[ 3, 5,-5, 1,-8],
[ 4, 3, 8, 2, 3]]);
print("(1)"); C:=rref(A); # leading ones in columns 1,2,4
BASIScolumnspace=col(A,1),col(A,2),col(A,4);
```

```

print("(2)"); F:=rref(transpose(A)); # leading ones in columns 1,2,3
      BASISrowSpace=row(A,1),row(A,2),row(A,3);
print("(3)"); nullspace(A); linsolve(A,vector([0,0,0,0]));
print("(4)"); RANK=rank(A); NULLITY=coldim(A)-rank(A);
print("(5)"); DIMnullspace=coldim(A)-rank(A); DIMrowSpace=rank(A);
      DIMcolumnSpace=rank(A);

```

**Example 3.** Let  $A = \begin{pmatrix} 1 & 1 & 1 & 2 & 6 \\ 2 & 3 & -2 & 1 & -3 \\ 3 & 5 & -5 & 1 & -8 \\ 4 & 3 & 8 & 2 & 3 \end{pmatrix}$ . Verify that the following column space bases of  $A$  are equivalent.

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 3 \\ 5 \\ 3 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 2 \end{pmatrix},$$

$$\mathbf{w}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -3 \end{pmatrix}, \quad \mathbf{w}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 17 \end{pmatrix}, \quad \mathbf{w}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -9 \end{pmatrix}.$$

**Solution:** We will use this result:

**Lemma 2.** Bases  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  and  $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$  are equivalent bases if and only if the augmented matrices  $F = \text{aug}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ ,  $G = \text{aug}(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)$  and  $H = \text{aug}(F, G)$  satisfy the rank condition  $\text{rank}(F) = \text{rank}(G) = \text{rank}(H) = 3$ .

The proof appears below.

The `maple` code which applies is

```

with(linalg):
A:=matrix([[ 1, 1, 1, 2, 6],
           [ 2, 3,-2, 1,-3],
           [ 3, 5,-5, 1,-8],
           [ 4, 3, 8, 2, 3]]);
v1:=vector([1,2,3,4]); v2:=vector([1,3,5,3]); v3:=vector([2,1,1,2]);
w1:=vector([1, 0, 0, -3]); w2:=vector([0, 1, 0, 17]); w3:=vector([0, 0, 1, -9]);
F:=augment(v1,v2,v3);
G:=augment(w1,w2,w3);
H:=augment(F,G);
rank(F); rank(G); rank(H);

```

We remark that the two bases in the example were discovered from the `maple` code

```

rref(A); # pivot cols 1,2,4
v1:=col(A,1); v2:=col(A,2); v3:=col(A,4);
B:=rref(transpose(A)); # pivot cols 1,2,3
w1:=row(B,1); w2:=row(B,2); w3:=row(B,3);

```

### Proof of Lemma 2.

**Proof:** The test appears in the online pdf documents at the course web site. Let's justify the test here, independently, showing only half the proof:  $\text{rank}(F) = \text{rank}(G) = \text{rank}(H) = n$  implies the bases are equivalent.

The equation  $\mathbf{rref}(F) = EF$  holds for  $E$  a product of elementary matrices. Then  $EH$  has to have  $n$  lead variables, because of  $F$  in the first  $n$  columns, and the remaining rows are zero, because  $\text{rank}(H) = n$ . Therefore, the first  $n$  columns of  $H$  are the pivot columns of  $H$ . The non-pivots of  $H$  are just the columns of  $G$ , and by the pivot theory, they are linear combinations of the pivot columns, namely, the columns of  $F$ . We can multiply  $H$  by a permutation matrix  $P$  which effectively swaps  $F$  and  $G$ . Already,  $HP$  has the  $n$  independent columns of  $F$ , so its rank is at least  $n$ . But its other columns are linear combinations of these columns, so the rank is exactly  $n$ . Now we argue by symmetry that the columns of  $F$  are linear combinations of the columns of  $G$ , using  $HP$  instead of  $H$ .

The proof is complete.

**Example 4.** Let  $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 0 & 0 \end{pmatrix}$ . Solve the equation  $A\mathbf{x} = -3\mathbf{x}$  for  $\mathbf{x}$ .

**Solution.** Let  $\lambda = -3$ . The idea is to write the equation  $A\mathbf{x} = \lambda\mathbf{x}$  as a homogeneous problem  $(A - \lambda I)\mathbf{x} = \mathbf{0}$ . Define  $B = A - \lambda I$ . The homogeneous equation  $B\mathbf{x} = \mathbf{0}$  always has the solution  $\mathbf{x} = \mathbf{0}$ . It has a nonzero solution  $\mathbf{x}$  if and only if there are infinitely many solutions, in which case the solutions are found by a frame sequence to  $\mathbf{rref}(B)$ . The **maple** details appear below. The basis vectors for  $B\mathbf{x} = \mathbf{0}$  are obtained in the usual way, by taking partial derivatives on the general solution with respect to the symbols  $t_1, t_2, \dots$ . In this case, there is just one basis vector

$$\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}.$$

```
with(linalg):
A:=matrix([[1,2,3],[2,-1,1],[3,0,0]]);
B:=evalm(A-(-3)*diag(1,1,1));
linsolve(B,vector([0,0,0]));
# ans: t_1*vector([-2,1,2])
# Basis == partial on t_1 == vector([-2,1,2])
```

**Proof of Lemma 1.** Define  $r_1 = 1, r_2 = -2, r_3 = 5$ . Assume  $AP = PT$ ,  $P = \mathbf{aug}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$  and  $T = \mathbf{diag}(r_1, r_2, r_3)$ . The definition of matrix multiplication implies that  $AP = \mathbf{aug}(A\mathbf{v}_1, A\mathbf{v}_2, A\mathbf{v}_3)$  and  $PT = \mathbf{aug}(r_1\mathbf{v}_1, r_2\mathbf{v}_2, r_3\mathbf{v}_3)$ . Then  $AP = PT$  holds if and only if the columns of the two matrices match, which is equivalent to the three equations  $A\mathbf{v}_1 = r_1\mathbf{v}_1, A\mathbf{v}_2 = r_2\mathbf{v}_2, A\mathbf{v}_3 = r_3\mathbf{v}_3$ . The proof is complete.

**End of Maple Lab 5 Linear Algebra.**