KEY

Differential Equations and Linear Algebra 2250-2

Midterm Exam 3, Spring 2007

Version 2: 10:45

Calculators, books, notes and computers are not allowed. Answer checks are not expected or required. First drafts are expected, not complete presentations. The midterm exam has 5 problems, some with multiple parts, suitable for 50 minutes.

Name KEY Classtime____ Midterm Exam 3 – Ver 2: 10:45

1. (ch4) Complete enough of the following to add to 100%.

(a) [100%] Let V be the vector space of all continuous functions defined on $0 \le x \le 1$. Define S to be the set of all continuously differentiable functions $f(x)$ in V such that $f'(1) = f(0) + \int_0^1 x f(x^2)dx$. Prove that S is a subspace of V, by using the Subspace Criterion.

(b) [30%] If you solved (a), then skip (b) and (c). Let V be the set of all 4×1 column vectors $\mathbf{x}$ with components x_1, x_2, x_3, x_4. Assume the usual $\mathcal{R}^4$ rules for addition and scalar multiplication. Let S be the subset of V defined by the equations $C\mathbf{x} = 0$, $\mathbf{b}\cdot\mathbf{x} = 0$, where C is a 2×4 matrix and $\mathbf{b}$ is a nonzero vector in V. Prove that S is a subspace of V.

(c) [70%] If you solved (a), then skip (b) and (c). Solve for the unknowns x_1, x_2, x_3, x_4 in the system of equations below by showing all details of a frame sequence from the augmented matrix C to $\mathbf{rref}(C)$. Report the **vector form** of the general solution.

$$\begin{array}{ccccccccc} x_1 & + & 2x_2 & - & 2x_3 & + & 3x_4 & = & 1 \\ & + & x_2 & + & 2x_3 & + & & = & 1 \\ x_1 & + & 3x_2 & + & & + & 3x_4 & = & 2 \\ x_1 & + & 4x_2 & + & 2x_3 & + & 3x_4 & = & 3 \end{array}$$

(a) $\vec{0}$ in S: Define $\vec{0}$ by equation $f(x)=0$. Then $f'(1)=0$ and $f(0)+\int_0^1 x f(x^2)dx = 0+0=0$ implies $\vec{0}$ in S.

$c_1\vec{v}_1+c_2\vec{v}_2$ in S: Define $\vec{v}_1, \vec{v}_2$ by eqs f_1, f_2. Define $\vec{v}=c_1\vec{v}_1+c_2\vec{v}_2$. Then $f=c_1f_1+c_2f_2$ is the eq for $\vec{v}$.

$$\begin{aligned} f'(1) &= c_1 f_1'(1) + c_2 f_2'(1) = c_1\left(f_1(0)+\int_0^1 x f_1(x^2)dx\right)+c_2\left(f_2(0)+\int_0^1 x f_2(x^2)dx\right) \\ &= \left(c_1 f_1(0)+c_2 f_2(0)\right) + \int_0^1 x\left(c_1 f_1(x^2)+c_2 f_2(x^2)\right)dx \\ &= f(0) + \int_0^1 x f(x)dx \end{aligned}$$

Proof complete by the subspace criterion.

(b) Define $A = \text{aug}(\text{row}(C,1), \text{row}(C,2), b^T)$. Then $A\vec{x}=\vec{0} \Leftrightarrow C\vec{x}=\vec{0}$ and $b^T\vec{x}=0$ (or $b\cdot\vec{x}=0$). Apply the kernel Theorem. Then S is a subspace of $\mathbb{R}^4$.

(c) $$A=\left(\begin{array}{cccc|c} 1 & 2 & -2 & 3 & 1 \\ 0 & 1 & 2 & 0 & 1 \\ 1 & 3 & 0 & 3 & 2 \\ 1 & 4 & 2 & 3 & 3 \end{array}\right) \quad \text{rref}(A)=\left(\begin{array}{cccc|c} 1 & 0 & -6 & 3 & 1 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left\{\begin{array}{l} x_1 - 6x_3 + 3x_4 = 1 \\ x_2 + 2x_3 = 0 \\ 0 = 0 \\ 0 = 0 \end{array}\right.$$

Reduced Echelon Sys

$$\left\{\begin{array}{l} x_1 = 1 + 6t_1 - 3t_2 \\ x_2 = -3t_1 \\ x_3 = t_1 \\ x_4 = t_2 \end{array}\right.$$

General Solution

$$\boxed{\vec{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t_1\begin{pmatrix} 6 \\ -3 \\ 1 \\ 0 \end{pmatrix} + t_2\begin{pmatrix} -3 \\ 0 \\ 0 \\ 1 \end{pmatrix}}$$

vector general solution

Start your solution on this page. Staple on extra pages as needed.

Name KEY Classtime_____ Midterm Exam 3 – Ver 2: 10:45

2. **(ch5)** Complete (a), (b) and then either (c) or (d). Do not do both (c) and (d).

(a) [30%] Given $8x''(t) + 26x'(t) + 21x(t) = 0$, which represents a damped spring-mass system with $m = 8$, $c = 26$, $k = 21$, **solve** the differential equation [20%] and **classify** the answer as over-damped, critically damped or under-damped [10%].

(b) [20%] Find a particular solution $y_p(t)$ and the homogeneous solution $y_h(x)$ for $y^{iv} + 4y'' = 10$.

(c) [50%] Find by undetermined coefficients the steady-state periodic solution for the equation $x'' + 4x' + 5x = 3\sin(2t)$.

(d) [50%] If you did (c) above, then skip this one! Find by variation of parameters a particular solution x_p for the equation $x'' + 4x' + 5x = e^{t^3}\tan(t^2)$. To save time, don't try to evaluate integrals (it's impossible).

(a) $(2r+3)(4r+7)=0$ $\quad x(t) = c_1 e^{-3t/2} + c_2 e^{-7t/4}$ $\quad$ over-damped

(b) $r^2(r^2+4)=0$ $\quad$ trial sol $y = d_1 x^2$ by the fixup rule

$y_p = \frac{10}{4}\frac{x^2}{2} = \frac{5x^2}{4}$ $\quad x_h = c_1 + c_2 x + c_3 \cos 2x + c_4 \sin 2x$

$y = y_h + y_p$

(c) $x_{ss} = \frac{-24}{65}\cos(2t) + \frac{3}{65}\sin(2t)$. Define $x = d_1 \cos 2t + d_2 \sin 2t$.

No fixup rule. Get equations $\begin{cases} d_1 + 8d_2 = 0 \\ -8d_1 + d_2 = 3 \end{cases}$. Solve, $d_1 = \frac{-24}{65}$, $d_2 = \frac{3}{65}$.

(d) $(r+2)^2+1=0$. atoms $= e^{-2t}\cos t, e^{-2t}\sin t$. $W = e^{-4t}$, $x_1 =$ First atom, $x_2 =$ second atom, $f = e^{t^3}\tan(t^2)$.

$$x_p = \left(-\int \frac{x_2 f}{W}\right)x_1 + \left(\int \frac{x_1 f}{W}\right)x_2$$

$$x_p = \left(-\int e^{2t}\sin t\, e^{t^3}\tan(t^2)\,dt\right)e^{-2t}\cos t + \left(\int e^{2t}\cos t\, e^{t^3}\tan(t^2)\,dt\right)e^{-2t}\sin t$$

Start your solution on this page. Staple on extra pages as needed.

Name KEY Classtime_____ Midterm Exam 3 – Ver 2: 10:45

3. **(ch5)** Complete all parts below.

(a) [70%] A non-homogeneous linear differential equation with constant coefficients has right side $f(x) = xe^x + 2(x+1)(x^2+2) + x\cos 2x$ and characteristic equation $r^2(r-1)^3(r^2+4) = 0$. Determine the **corrected** trial solution for y_p according to the method of undetermined coefficients and the **fixup rule**. To save time, **do not** evaluate the undetermined coefficients and **do not** find $y_p(x)$! Undocumented detail or guessing earns no credit.

(b) [30%] Write out the general solution of the homogeneous linear constant coefficient equation whose characteristic equation is $(r^3+r)^2(r^2+r)^3(r^2+4r)^3 = 0$.

(a) atoms of $f = 1, x, x^2, x^3, e^x, xe^x, \cos 2x, \sin 2x, x\cos 2x, x\sin 2x$

Char eq roots $= 0, 0, 1, 1, 1, 2i, -2i$

$y = x^2(d_1 + d_2 x + d_3 x^2 + d_4 x^3)$ atomRoot $= 0$

$+ x^3(d_5 e^x + d_6 xe^x)$ atomRoot $= 1$

$+ x(d_7 \cos 2x + d_8 \sin 2x + d_9 x\cos 2x + d_{10} x\sin 2x)$ atomRoot$=2i$

(b) $r^2(r^2+1)^2 r^3(r+1)^3 r^3(r+4)^3 = 0$

roots $= 0,0,0,0,0,0,0,0, i,-i,i,-i, -1,-1,-1, -4,-4,-4$

$y = c_1 + c_2 x + c_3 x^2 + c_4 x^3 + c_5 x^4 + c_6 x^5 + c_7 x^6 + c_8 x^7$

$+ (c_9 + c_{10} x)\cos x + (c_{11} + c_{12} x)\sin x$

$+ (c_{13} + c_{14} x + c_{15} x^2)e^{-x}$

$+ (c_{16} + c_{17} x + c_{18} x^2)e^{-4x}$

Start your solution on this page. Staple on extra pages as needed.

Name KEY Classtime____ Midterm Exam 3 – Ver 2: 10:45

4. (ch6) Complete all of the items below.

(a) [30%] Find the eigenvalues of the matrix $A = \begin{pmatrix} 4 & -2 & 1 & 12 \\ 2 & 4 & -3 & 15 \\ 0 & 0 & 3 & 7 \\ 0 & 0 & -1 & -5 \end{pmatrix}$. To save time, **do not** find eigenvectors!

(b) [40%] Given $A = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix}$, assume there exists an invertible matrix P and a diagonal matrix D such that $AP = PD$. Circle all possible columns of P from the list below.

$$\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}.$$

(c) [30%] Find all eigenpairs for the matrix $A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$. Expect complex numbers in the answers.

(a) Cofactor expansion gives $\det(A - \lambda I) =$ polynomial of degree 4.
The roots are $\lambda = 2, -4, 4 \pm 2i$

(b) $\begin{pmatrix} 1 & 1 & -1 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 4 \end{pmatrix} \neq \lambda \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ for any λ. Not an eigenvector.
The others work.

(c) $\det(A - \lambda I) = (1-\lambda)^2 + 4$. Roots $1 \pm 2i$

$B = A - (1+2i)I$

$= \begin{pmatrix} -2i & 2 \\ -2 & -2i \end{pmatrix}$

$\begin{pmatrix} -i & 1 \\ 0 & 0 \end{pmatrix}$

$\begin{pmatrix} 1 & i \\ 0 & 0 \end{pmatrix}$

$\begin{cases} x_1 + i x_2 = 0 \\ 0 = 0 \end{cases}$

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = t_1 \begin{pmatrix} -i \\ 1 \end{pmatrix}$

Choose $\vec{V}_1 = \frac{\partial}{\partial t_1} = \begin{pmatrix} -i \\ 1 \end{pmatrix}$

$\vec{V}_2 =$ conjugate of $\vec{V}_1 = \begin{pmatrix} i \\ 1 \end{pmatrix}$

$\boxed{\text{Eigenpairs} = \left(2i, \begin{pmatrix} -i \\ 1 \end{pmatrix}\right), \left(-2i, \begin{pmatrix} i \\ 1 \end{pmatrix}\right)}$

Start your solution on this page. Staple on extra pages as needed.

Name KEY Classtime_____ Midterm Exam 3 – Ver 2: 10:45

5. **(ch6)** Complete all parts below.

Consider the 3×3 matrix

$$A = \begin{pmatrix} 4 & 2 & -2 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix}.$$

Already computed are eigenpairs

$$\left(2, \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}\right), \quad \left(4, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right).$$

(a) [25%] Find the remaining eigenpairs of A.

(b) [25%] Display an invertible matrix P and a diagonal matrix D such that $AP = PD$.

(c) [25%] Display explicitly Fourier's model for A.

(d) [25%] Display the vector general solution $\mathbf{x}(t)$ of the linear differential system $\mathbf{x}' = A\mathbf{x}$.

(a) $\det(A - \lambda I) = (4-\lambda)((3-\lambda)^2 - 1) = (4-\lambda)(2-\lambda)(4-\lambda)$

$B = A - 4I = \begin{pmatrix} 0 & 2 & -2 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$

$\begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ rref(B)

$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$\boxed{\text{Last eigenpair} = \left(4, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right)}$

(b) $D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ $\quad P = \begin{pmatrix} 2 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$

(c) $A\left(c_1 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right) = c_1(2)\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + c_2(4)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_3(4)\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

(d) $\vec{x}(t) = c_1 e^{2t}\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + c_2 e^{4t}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_3 e^{4t}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

Start your solution on this page. Staple on extra pages as needed.