KEY

Differential Equations and Linear Algebra 2250-1

Midterm Exam 3, Spring 2007

Version 1: 7:30

Calculators, books, notes and computers are not allowed. Answer checks are not expected or required. First drafts are expected, not complete presentations. The midterm exam has 5 problems, some with multiple parts, suitable for 50 minutes.

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1. **(ch4)** Complete enough of the following to add to 100%.

(a) [100%] Let V be the vector space of all continuous functions defined on $0 \le x \le 1$. Define S to be the set of all continuously differentiable functions $f(x)$ in V such that $f(1) = f(0) + \int_0^1 x^2 f(x)dx$. Prove that S is a subspace of V, by using the Subspace Criterion.

(b) [30%] If you solved (a), then skip (b) and (c). Let V be the set of all 3×1 column vectors $\mathbf{x}$ with components x_1, x_2, x_3. Assume the usual $\mathcal{R}^3$ rules for addition and scalar multiplication. Let S be the subset of V defined by the equations $C\mathbf{x} = 0$, $\mathbf{b} \cdot \mathbf{x} = 0$, where C is a 2×3 matrix and $\mathbf{b}$ is a nonzero vector in V. Prove that S is a subspace of V.

(c) [70%] If you solved (a), then skip (b) and (c). Solve for the unknowns x_1, x_2, x_3, x_4 in the system of equations below by showing all details of a frame sequence from the augmented matrix C to **rref**(C). Report the **vector form** of the general solution.

$$\begin{array}{ccccccccc} x_1 & + & 4x_2 & - & 2x_3 & + & 3x_4 & = & 1 \\ & + & 2x_2 & + & 2x_3 & + & & = & 1 \\ x_1 & + & 6x_2 & + & & + & 3x_4 & = & 2 \\ x_1 & + & 8x_2 & + & 2x_3 & + & 3x_4 & = & 3 \end{array}$$

(a) Let $\vec{v} = \vec{0}$. Then $\vec{v}$ is the equation $f(x) = 0$. It satisfies $f(1) = f(0) + \int_0^1 x^2 f(x)dx$, so $\vec{v} = \vec{0}$ is in S.

Let c_1, c_2 be scalars and $\vec{v}_1, \vec{v}_2$ be in S. Then $\vec{v}_1, \vec{v}_2$ are given by f_1, f_2, and $f_1(1) = f_1(0) + \int_0^1 x^2 f_1(x)dx$, $f_2(1) = f_2(0) + \int_0^1 x^2 f_2(x)dx$. Multiply each equation by c_1, c_2, resp. Then add to get for $f = c_1 f_1 + c_2 f_2$ the relation

$$f(1) = f(0) + \int_0^1 \left(x^2 f_1(x) + x^2 f_2(x)\right) dx$$
$$= f(0) + \int_0^1 x^2 f(x) dx$$

Then $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2$ is represented by f and $\vec{v}$ is in S. This completes the application of the Subspace Criterion. Then S is a subspace of V. The proof is complete.

(b) Apply the kernel Thm: $S = \{\vec{x} \text{ in } V : A\vec{x} = \vec{0}\} \Rightarrow S = $ subspace of V.
Let A have the same rows as C and last row b^T. Then $A\vec{x} = \vec{0}$ $\Leftrightarrow$ $C\vec{x} = \vec{0}$ and $b \cdot \vec{x} = 0$. Then $S =$ a subspace of V, by the kernel Thm.

(c) $C = \left(\begin{array}{cccc|c} 1 & 4 & -2 & 3 & 1 \\ 0 & 2 & 2 & 0 & 1 \\ 1 & 6 & 0 & 3 & 2 \\ 1 & 8 & 2 & 3 & 3 \end{array}\right)$ $\quad$ $\text{rref}(C) = \left(\begin{array}{cccc|c} 1 & 0 & -6 & 3 & -1 \\ 0 & 1 & 1 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$ $\quad$ Two free vars, x_3, x_4.

$$\begin{cases} x_1 = 6t_1 - 3t_2 - 1 \\ x_2 = -t_1 + 1/2 \\ x_3 = t_1 \\ x_4 = t_2 \end{cases}$$

$$\boxed{\vec{x} = \begin{pmatrix} -1 \\ 1/2 \\ 0 \\ 0 \end{pmatrix} + t_1 \begin{pmatrix} 6 \\ -1 \\ 1 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} -3 \\ 0 \\ 0 \\ 1 \end{pmatrix}}$$

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2. **(ch5)** Complete (a), (b) and then either (c) or (d). **Do not do both** (c) and (d).

(a) [30%] Given $2x''(t) + 23x'(t) + 45x(t) = 0$, which represents a damped spring-mass system with $m = 2$, $c = 23$, $k = 45$, **solve** the differential equation [20%] and **classify** the answer as over-damped, critically damped or under-damped [10%].

(b) [20%] Find a particular solution $y_p(t)$ and the homogeneous solution $y_h(x)$ for $y^{iv} + 16y'' = 17$.

(c) [50%] Find by undetermined coefficients the steady-state periodic solution for the equation $x'' + 2x' + 17x = \sin(t)$.

(d) [50%] If you did (c) above, then skip this one! Find by variation of parameters a particular solution x_p for the equation $x'' + 4x' + 20x = e^{t^2}\tan(t^3)$. To save time, don't try to evaluate integrals (it's impossible).

(a) $(2r+5)(r+9) = 0 \qquad x(t) = c_1 e^{-5t/2} + c_2 e^{-9t}$ over-damped

(b) $y_p = \frac{17}{16}\left(\frac{x^2}{2}\right) \qquad y_h = c_1 + c_2 x + c_3 \cos 4x + c_4 \sin 4x$

(c) trial sol $x = d_1 \cos t + d_2 \sin t$. No fixup rule needed.

$$\begin{cases} 16d_1 + 2d_2 = 0 \\ -2d_1 + 16d_2 = 1 \end{cases} \Rightarrow d_1 = \frac{-1}{130},\ d_2 = \frac{4}{65} \text{ by Cramer's Rule.}$$

$$\boxed{x(t) = -\frac{\cos t + 8\sin t}{130}}$$

(d) $(r+2)^2 + 16 = 0 \Rightarrow r = -2 \pm 4i \Rightarrow x_1 = e^{-2t}\cos 4t,\ x_2 = e^{-2t}\sin 4t$

$W = \begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix} = 4e^{-4t} \qquad f(t) = e^{t^2}\tan(t^3)$

$$x_p(t) = \left(-\int \frac{x_2 f}{W}dt\right)x_1 + \left(\int \frac{x_1 f}{W}dt\right)x_2$$

$$= \left(-\int \frac{e^{-2t}\sin 4t\, e^{t^2}\tan(t^3)\,dt}{4e^{-4t}}\right)e^{-2t}\cos 4t$$

$$+ \left(\int \frac{e^{-2t}\cos 4t\, e^{t^2}\tan(t^3)\,dt}{4e^{-4t}}\right)e^{-2t}\sin 4t$$

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3. **(ch5)** Complete all parts below.

(a) [70%] A non-homogeneous linear differential equation with constant coefficients has right side $f(x) = x^2e^{-x} + 2(x+1)(x^2+4) + x\sin 4x$ and characteristic equation $r^2(r+1)^3(r^2+16) = 0$. Determine the **corrected** trial solution for y_p according to the method of undetermined coefficients and the **fixup rule**. To save time, **do not** evaluate the undetermined coefficients and **do not** find $y_p(x)$! Undocumented detail or guessing earns no credit.

(b) [30%] Write out the general solution of the homogeneous linear constant coefficient equation whose characteristic equation is $(r^3 - r)(r^2 - r)^2(r^2 + 4r)^2 = 0$.

(a) atoms of f = $1, x, x^2, x^3$, $e^{-x}, xe^{-x}, x^2e^{-x}$, $\cos 4x, \sin 4x$, $x\cos 4x, x\sin 4x$

root list of char eq = $0, 0, -1, -1, -1, 4i, -4i$

Trial sol $y = x^2(d_1 + d_2 x + d_3 x^2 + d_4 x^3)$
$+ x^3(d_5 + d_6 x + d_7 x^2)e^{-x}$
$+ x(d_8 \cos 4x + d_9 \sin 4x + d_{10} x\cos 4x + d_{11} x \sin 4x)$

because atomRoot = $0, -1, 4i$ for the 3 groups (one group per line).

(b) roots of char eq = $0,0,0,0,0,\ 1,1,1,\ -1,\ -4,-4$

atoms = $1, x, x^2, x^3, x^4$, e^x, xe^x, x^2e^x, e^{-x}, e^{-4x}, xe^{-4x}

$y_h = c_1 + c_2 x + c_3 x^2 + c_4 x^3 + c_5 x^4$
$+ (c_6 + c_7 x + c_8 x^2)e^x$
$+ c_9 e^{-x}$
$+ (c_{10} + c_{11} x)e^{-4x}$

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4. **(ch6)** Complete all of the items below.

(a) [30%] Find the eigenvalues of the matrix $A = \begin{pmatrix} 3 & -2 & 1 & 12 \\ 2 & 3 & -3 & 15 \\ 0 & 0 & 2 & 7 \\ 0 & 0 & -1 & -6 \end{pmatrix}$. To save time, **do not** find eigenvectors!

(b) [40%] Given $A = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix}$, assume there exists an invertible matrix P and a diagonal matrix D such that $AP = PD$. Circle all possible columns of P from the list below.

$$\boxed{\begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -5 \\ -5 \end{pmatrix}.}$$

(c) [30%] Find all eigenpairs for the matrix $A = \begin{pmatrix} 1 & 7 \\ -7 & 1 \end{pmatrix}$. Expect complex numbers in the answers.

(a) $\left((3-\lambda)^2+4\right)(\lambda+5)(\lambda-1)=0$ $\boxed{\lambda = 1, -5, 3+2i, 3-2i}$

(b) $A\begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} = 2\begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$

Because all listed vectors are eigenvectors, all are possible columns of P.

$A\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$A\begin{pmatrix} 0 \\ -5 \\ -5 \end{pmatrix} = \begin{pmatrix} 0 \\ -20 \\ -20 \end{pmatrix} = 4\begin{pmatrix} 0 \\ -5 \\ -5 \end{pmatrix}$

(c) $(1-\lambda)^2 + 7^2 = 0$, $\boxed{\lambda_1 = 1+7i, \lambda_2 = 1-7i}$

$B = A-(1+7i)I = \begin{pmatrix} -7i & 7 \\ -7 & -7i \end{pmatrix}$ $\quad \text{rref}(B) = \begin{pmatrix} 1 & i \\ 0 & 0 \end{pmatrix}$

$\begin{cases} x_1 = -i t_1 \\ x_2 = t_1 \end{cases}$ $\boxed{\vec{v}_1 = \begin{pmatrix} -i \\ 1 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} i \\ 1 \end{pmatrix}}$

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5. **(ch6)** Complete all parts below.

Consider the 3×3 matrix

$$A = \begin{pmatrix} 5 & 2 & -2 \\ 0 & 4 & 1 \\ 0 & 1 & 4 \end{pmatrix}.$$

Already computed are eigenpairs

$$\left(3, \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}\right), \quad \left(5, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right).$$

(a) [25%] Find the remaining eigenpairs of A.

(b) [25%] Display an invertible matrix P and a diagonal matrix D such that $AP = PD$.

(c) [25%] Display explicitly Fourier's model for A.

(d) [25%] Display the vector general solution $\mathbf{x}(t)$ of the linear differential system $\mathbf{x}' = A\mathbf{x}$.

(a) $(5-\lambda)(\lambda^2 - 8\lambda + 15) = 0$ $\quad\lambda_1 = 3,\ \lambda_2 = 5$ (double root)

$(5-\lambda)(\lambda-3)(\lambda-5) = 0$

$B = A - \lambda_2 I = \begin{pmatrix} 0 & 2 & -2 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$ $\quad \mathrm{rref}(B) = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$\boxed{\left(5, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right) = 3^{\text{rd}} \text{ eigenpair}}$

(b) $P = \begin{pmatrix} 2 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ $\quad D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}$

(c) $A\left(c_1 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right) = 3c_1 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + 5c_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 5c_3 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

(d) $\vec{x}(t) = c_1 e^{3t} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + c_2 e^{5t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_3 e^{5t} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

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