Calculators, books, notes and computers are not allowed. Answer checks are not expected or required. First drafts are expected, not complete presentations. The midterm exam has 5 problems, some with multiple parts, suitable for 50 minutes.
1. (ch4) Complete enough of the following to add to 100%.

(a) [100%] Let \( V \) be the vector space of all continuous functions defined on \( 0 \leq x \leq 1 \). Define \( S \) to be the set of all continuously differentiable functions \( f(x) \) in \( V \) such that \( f(1) = f(0) + \int_0^1 x^2 f(x) \, dx \). Prove that \( S \) is a subspace of \( V \), by using the Subspace Criterion.

(b) [30%] If you solved (a), then skip (b) and (c). Let \( V \) be the set of all \( 3 \times 1 \) column vectors \( \mathbf{x} \) with components \( x_1, x_2, x_3 \). Assume the usual \( \mathbb{R}^3 \) rules for addition and scalar multiplication. Let \( S \) be the subset of \( V \) defined by the equations \( \mathbf{C} \mathbf{x} = 0 \), \( \mathbf{b} \cdot \mathbf{x} = 0 \), where \( \mathbf{C} \) is a \( 2 \times 3 \) matrix and \( \mathbf{b} \) is a nonzero vector in \( V \). Prove that \( S \) is a subspace of \( V \).

(c) [70%] If you solved (a), then skip (b) and (c). Solve for the unknowns \( x_1, x_2, x_3, x_4 \) in the system of equations below by showing all details of a frame sequence from the augmented matrix \( \mathbf{C} \) to \( \text{rref}(\mathbf{C}) \).

\[
\begin{align*}
\begin{align*}
x_1 + 4x_2 - 2x_3 + 3x_4 &= 1 \\
+ 2x_2 + 2x_3 &= 1 \\
x_1 + 6x_2 &= 2 \\
x_1 + 8x_2 + 2x_3 + 3x_4 &= 3
\end{align*}
\end{align*}
\]

2. Let \( \mathbf{v} = 0 \). Then \( \mathbf{v} \) is the equation \( f(x) = 0 \). It satisfies \( f(1) = f(0) + \int_0^1 x^2 f(x) \, dx \), so \( \mathbf{v} = 0 \) is in \( S \).

Let \( c_1, c_2 \) be scalars and \( \mathbf{v}_1, \mathbf{v}_2 \) be in \( S \). Then \( \mathbf{v}_1, \mathbf{v}_2 \) are given by \( f_1, f_2 \), where \( f_1(1) = f_2(1) + \int_0^1 x^2 f_1(x) \, dx \). Multiply each equation by \( c_1, c_2 \), respectively, then add to get for \( \mathbf{v} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 \), \( \mathbf{v} \) is in \( S \).

Then \( \mathbf{v} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 \) is represented by \( f \) and \( \mathbf{v} \) is in \( S \). This completes the application of the Subspace Criterion. Then \( S \) is a subspace of \( V \). The proof is complete.

3. 

Apply the Kernel Theorem: \( S = \{ \mathbf{x} \in V : A \mathbf{x} = \mathbf{0} \} \Rightarrow S = \text{subspace of } V \).

Let \( A \) have the same rows as \( \mathbf{C} \) and last row \( \mathbf{b} \). Then \( A \mathbf{\tilde{x}} = \mathbf{0} \) \( \Leftrightarrow \mathbf{C} \mathbf{\tilde{x}} = \mathbf{0} \) and \( \mathbf{b} \cdot \mathbf{\tilde{x}} = 0 \). Then \( S \) is a subspace of \( V \), by the Kernel Theorem.

4. 

\[
\mathbf{C} = \begin{pmatrix}
1 & 4 & -2 & 3 \\
1 & 6 & 0 & 2 \\
1 & 8 & 2 & 3
\end{pmatrix}
\quad \text{rref}(\mathbf{C}) = \begin{pmatrix}
1 & 0 & -6 & 2 \\
0 & 1 & 0 & -1 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

Two free vars: \( x_3, x_4 \).

\[
\begin{align*}
x_1 &= 6t_1 - 3t_2 - 1 \\
x_2 &= -t_1 + t_2 \\
x_3 &= t_1 \\
x_4 &= t_2
\end{align*}
\]

\[
\mathbf{x} = \begin{pmatrix}
-1 \\
0 \\
0 \\
1
\end{pmatrix} + t_1 \begin{pmatrix}
-6 \\
1 \\
0 \\
-1
\end{pmatrix} + t_2 \begin{pmatrix}
2 \\
0 \\
0 \\
0
\end{pmatrix}
\]

Start your solution on this page. Staple on extra pages as needed.
2. (ch5) Complete (a), (b) and then either (c) or (d). Do not do both (c) and (d).

(a) [30%] Given \(2x''(t) + 23x'(t) + 45x(t) = 0\), which represents a damped spring-mass system with \(m = 2, c = 23, k = 45\), solve the differential equation [20%] and classify the answer as over-damped, critically damped or under-damped [10%].

(b) [20%] Find a particular solution \(y_p(t)\) and the homogeneous solution \(y_h(x)\) for \(y^{iv} + 16y'' = 17\).

(c) [50%] Find by undetermined coefficients the steady-state periodic solution for the equation \(x'' + 2x' + 17x = \sin(t)\).

(d) [50%] If you did (c) above, then skip this one! Find by variation of parameters a particular solution \(x_p\) for the equation \(x'' + 4x' + 20x = e^{t^2} \tan(t^3)\). To save time, don’t try to evaluate integrals (it’s impossible).

\[ (2r+5)(r+9) = 0 \quad x(t) = c_1 e^{-5t/2} + c_2 e^{-9t} \quad \text{over-damped} \]

\[ y_p = \frac{17}{16} \left( \frac{y^2}{2} \right) \quad y_h = c_1 + c_2 \sin x + c_3 \cos 4x + c_4 \sin 4x \]

(c) Trial sol \(x = d_1 \cos t + d_2 \sin t\). No fixup rule needed.

\[
\begin{align*}
16d_1 + 2d_2 &= 0 \\
-2d_1 + 16d_2 &= 1
\end{align*}
\Rightarrow d_1 = \frac{-1}{130}, d_2 = \frac{1}{5} \quad \text{By Cramer's Rule.}
\]

\[
x(t) = -\frac{\cos t + 8 \sin t}{130}
\]

(d) \((r+2)^2 + 16 = 0 \Rightarrow r = -2 \pm 4i \Rightarrow x_1 = e^{-2t} \cos 4t, x_2 = e^{-2t} \sin 4t
\]

\[
W = \begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix} = 4 e^{-4t}
\]

\[
f(t) = e^{t^2} \tan(t^3)
\]

\[
x_p(t) = \left(- \int \frac{x_2 f dt}{W} \right) x_1 + \left( \int \frac{x_1 f dt}{W} \right) x_2
\]

\[
= \left(- \int e^{2t} \sin 4t \frac{e^{t^2} \tan(t^3) dt}{4 e^{-4t}} \right) e^{-2t} \cos 4t
\]

\[
+ \left( \int e^{2t} \cos 4t \frac{e^{t^2} \tan(t^3) dt}{4 e^{-4t}} \right) e^{-2t} \sin 4t
\]

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3. (ch5) Complete all parts below.

(a) [70%] A non-homogeneous linear differential equation with constant coefficients has right side \( f(x) = x^2 e^{-x} + 2(x + 1)(x^2 + 4) + x \sin 4x \) and characteristic equation \( r^2(r + 1)^3(r^2 + 16) = 0 \). Determine the corrected trial solution for \( y_p \) according to the method of undetermined coefficients and the fixup rule. To save time, do not evaluate the undetermined coefficients and do not find \( y_p(x) \)! Undocumented detail or guessing earns no credit.

(b) [30%] Write out the general solution of the homogeneous linear constant coefficient equation whose characteristic equation is \( (r^3 - r)(r^2 - r)^2(r^2 + 4r)^2 = 0 \).

\[ a \] atoms \( R \) = \( 1, x, x^2, x^3, e^x, xe^{-x}, x e^{-x}, \cos 4x, \sin 4x, \cos 4x, \sin 4x \)

root list \( q \) char eq = \( 0, 0, -1, -1, -1, 4 \pm i, -4 i \)

Trial sol \( y = x^2 (d_1 + d_2 x + d_3 x^2 + d_4 x^3) 
+ x^3 (d_5 + d_6 x + d_7 x^2) e^{-x} 
+ x (d_8 \cos 4x + d_9 \sin 4x + d_{10} \cos 4x + d_{11} \sin 4x) \)

because atom \( \text{Root} = 0, -1, 4 \pm i \) for \( \text{The 3 groups (one group per line)} \).

\[ b \] roots \( q \) char eq = \( 0, 0, 0, 0, 0, 1, 1, 1, -1, -4, -4 \)

atoms = \( 1, x, x^2, x^3, x^4, e^x, xe^x, x e^{-x}, e^{-x}, e^{-4x}, xe^{-4x} \)

\( y_b = c_1 + c_2 x + c_3 x^2 + c_4 x^3 + c_5 x^4 
+ (c_6 + c_7 x + c_8 x^2) e^x 
+ c_9 e^{-x} 
+ (c_{10} + c_{11} x) e^{-4x} \)
4. (ch6) Complete all of the items below.

(a) [30%] Find the eigenvalues of the matrix $A = \begin{pmatrix} 3 & -2 & 1 & 12 \\ 2 & 3 & -3 & 15 \\ 0 & 0 & 2 & 7 \\ 0 & 0 & -1 & -6 \end{pmatrix}$. To save time, do not find eigenvectors!

(b) [40%] Given $A = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix}$, assume there exists an invertible matrix $P$ and a diagonal matrix $D$ such that $AP = PD$. Circle all possible columns of $P$ from the list below.

\[
\begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -5 \\ -5 \end{pmatrix}.
\]

(c) [30%] Find all eigenpairs for the matrix $A = \begin{pmatrix} 1 & 7 & \end{pmatrix}$. Expect complex numbers in the answers.

\[ (3-\lambda)^2 + 4 \right) (\lambda + 5)(\lambda - 1) = 0 \]

\[ \lambda = 1, -5, 3 + 2i, 3 - 2i \]

\[ \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} = 2 \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} \]

\[ A \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \]

\[ A \begin{pmatrix} 0 \\ -5 \\ -5 \end{pmatrix} = \begin{pmatrix} 0 \\ -20 \\ -20 \end{pmatrix} = 4 \begin{pmatrix} 0 \\ -5 \\ -5 \end{pmatrix} \]

\[ (1-\lambda)^2 + 7^2 = 0, \quad \lambda_1 = 1 + 7i, \lambda_2 = 1 - 7i \]

\[ B = A - (1 + 7i)I = \begin{pmatrix} -7i & 7 \\ -7 & -7i \end{pmatrix} \quad \text{ref}(B) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \]

\[ \begin{cases} x_1 = -i t_1 \\ x_2 = t_1 \end{cases} \quad \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} i \\ 1 \end{pmatrix} \]

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5. (ch6) Complete all parts below.

Consider the $3 \times 3$ matrix

$$A = \begin{pmatrix} 5 & 2 & -2 \\ 0 & 4 & 1 \\ 0 & 1 & 4 \end{pmatrix}. $$

Already computed are eigenpairs

$$\left(3, \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}\right), \quad \left(5, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right).$$

(a) [25%] Find the remaining eigenpairs of $A$.
(b) [25%] Display an invertible matrix $P$ and a diagonal matrix $D$ such that $AP = PD$.
(c) [25%] Display explicitly Fourier's model for $A$.
(d) [25%] Display the vector general solution $\mathbf{x}(t)$ of the linear differential system $\mathbf{x}' = A\mathbf{x}$.

\[\begin{aligned}
(a) \quad & (5-\lambda)\left(\lambda^2-8\lambda+15\right) = 0 \\
& \lambda_1 = 3, \quad \lambda_2 = 5 \quad \text{(double root)} \\
& B = A - \lambda_2 I = \begin{pmatrix} 0 & 2 & -2 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \\
& \text{ref}(B) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
& \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\
& \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\
& (5, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}) = 3 \times 2 \text{ eigenpair}
\end{aligned}\]

\[\begin{aligned}
(b) \quad & P = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \\
& D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\end{aligned}\]

\[\begin{aligned}
(c) \quad & A \begin{pmatrix} c_1 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \end{pmatrix} = 3c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 5c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 5c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
\end{aligned}\]

\[\begin{aligned}
(d) \quad & \mathbf{x}(t) = c_1 e^{3t} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + c_2 e^{5t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + c_3 e^{5t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}
\end{aligned}\]

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