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Applied Differential Equations 2250

Exam date: Tuesday, 13 March, 2007

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%.

1. **(Frame sequences and the 3 properties)**
Determine a, b such that the system has a unique solution, infinitely many solutions, or no solution:

$$\begin{array}{rcrcrcl} x &+& 2y &+& z &=& a \\ 3x &+& 2by &+& 2z &=& b \\ 4x &+& 8y &+& 3z &=& 2+a \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & a \\ 3 & 2b & 2 & b \\ 4 & 8 & 3 & 2+a \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & a \\ 0 & 2b-6 & -1 & b-3a \\ 4 & 8 & 3 & 2+a \end{array}\right) \text{combo}(1,2,-3)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & a \\ 0 & 2b-6 & -1 & b-3a \\ 0 & 0 & -1 & 2-3a \end{array}\right) \text{combo}(1,3,-4)$$

case $2b-6=0$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & a \\ 0 & 0 & -1 & 3-3a \\ 0 & 0 & -1 & 2-3a \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & a \\ 0 & 0 & -1 & 3-3a \\ 0 & 0 & 0 & -1 \end{array}\right)$$

Signal eq $0=-1$

$\boxed{\text{No sol for } b=3}$

case $2b-6\neq 0$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & a \\ 0 & 1 & x & y \\ 0 & 0 & 1 & 3a-2 \end{array}\right) \quad x = \frac{-1}{2b-6} \quad y = \frac{b-3a}{2b-6}$$

Three leading ones

$\boxed{\text{Unique sol for } b\neq 3}$

$\boxed{\text{Never has } \infty\text{-many sols}}$

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2. **(vector spaces)** Do all three parts.

(a) [20%] The vector space V is the set of all matrices $\begin{pmatrix} a & b \\ -a & a+b \end{pmatrix}$. Display a basis for V. Don't justify anything.

(b) [40%] Let V be the vector space of all column vectors $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and let S be the subset of V given by the equations $x_1 = 0$, $x_2 + 2x_3 = 0$. Prove that S is a subspace of V.

(c) [40%] Find a basis of vectors for the subspace of $\mathcal{R}^4$ given by the system of equations

$$\begin{aligned} x_1 + 4x_2 - 2x_3 + x_4 &= 0, \\ x_1 + 2x_2 - 3x_3 + x_4 &= 0, \\ 4x_2 + 2x_3 &= 0. \end{aligned}$$

(a) $\{\partial_a, \partial_b\}$ = Basis $\Rightarrow$ Basis $= \left\{ \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \right\}$

(b) Define $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$. Then $S = \{\vec{x} : A\vec{x} = \vec{0}\}$ is a subspace by Thm 2 (kernel Theorem).

(c) $\begin{pmatrix} 1 & 4 & -2 & 1 \\ 1 & 2 & -3 & 1 \\ 0 & 4 & 2 & 0 \end{pmatrix}$

$\begin{pmatrix} 1 & 4 & -2 & 1 \\ 0 & -2 & -1 & 0 \\ 0 & 4 & 2 & 0 \end{pmatrix}$ combo(1,2,-1)

$\begin{pmatrix} 1 & 4 & -2 & 1 \\ 0 & -2 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ combo(2,3,2)

$\begin{pmatrix} 1 & 0 & -4 & 1 \\ 0 & -2 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ combo(2,1,2)

$\begin{pmatrix} 1 & 0 & -4 & 1 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ mult(2,-1/2)

$\begin{cases} x_1 - 4x_3 + x_4 = 0 \\ x_2 + x_3/2 = 0 \\ 0 = 0 \end{cases}$

$\begin{cases} x_1 = 4t_1 - t_2 \\ x_2 = -t_1/2 \\ x_3 = t_1 \\ x_4 = t_2 \end{cases}$ Scalar general Solution

Basis $= \{\partial_{t_1}, \partial_{t_2}\}$

$\boxed{\text{Basis} = \left\{ \begin{pmatrix} 4 \\ -1/2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}}$

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3. (independence) Do **only two** of the three parts.

(a) [50%] Let $\mathbf{u}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{u}_3 = \begin{pmatrix} 1 \\ 1 \\ 3 \\ 3 \\ 1 \end{pmatrix}$. State a test that decides independence or dependence of the list of three vectors [20%]. Apply the test and report the result [30%].

(b) [50%] State the pivot theorem. Then extract from the list below a largest set of independent vectors.

$\mathbf{a} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2 \\ -2 \\ 0 \\ 2 \end{pmatrix}$, $\mathbf{c} = \begin{pmatrix} 3 \\ -1 \\ 0 \\ 5 \end{pmatrix}$, $\mathbf{d} = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 2 \end{pmatrix}$, $\mathbf{e} = \begin{pmatrix} 2 \\ 2 \\ 0 \\ 6 \end{pmatrix}$,

(c) [50%] [If you did (a) and (b) already, then 100% has been attained: skip this one!]
Assume that columns 1 and 2 of the 2×3 matrix D are pivots. Prove that for some constants a and b, $\mathbf{rref}(D) = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \end{pmatrix}$.

(a) Vectors u_1, u_2, u_3 in $\mathbb{R}^5$ are independent $\Leftrightarrow$ rank(A) = 3 where A = augmented matrix of u_1, u_2, u_3.

$A = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & 3 \\ 1 & 1 & 3 \\ 1 & 0 & 1 \end{pmatrix}$ $\quad \text{rref}(A) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad$ rank(A) = 2 $\Rightarrow$ u_1, u_2, u_3 are dependent

(b) pivot Theorem
1. The pivot columns of a matrix A are independent.
2. Non-pivot columns are linear combinations of the pivot columns of A.

$A = \text{aug}(a, b, c, d, e)$ $\quad \text{rref}(A) = \begin{pmatrix} 1 & 2 & 0 & -3 & -4 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ $\quad \boxed{\vec{a}, \vec{c}}$

(c) rank(D) is 2 and the two leading ones must appear in columns 1, 2 of rref(D). These columns are those of $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ in natural order. Then $D = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \end{pmatrix}$ for some constants a, b.

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4. (determinants and elementary matrices) Do both parts.

(a) [50%] Assume given 3×3 matrices A, B. Suppose $A = E_3E_2E_1B$ and E_1, E_2, E_3 are elementary matrices representing respectively a swap, a combination and a multiply by -4. Assume $\det(A) = \sqrt{5}$. Find $\det(3A^2B)$.

(b) [50%] Let A, B and C be 4×4 matrices such that $C + 4B^2 + 2BA = A^2 + 2AB$. Suppose C contains a pair of duplicate columns and $\det(A + 2B) = -5$. Find the value of $\det(A - 2B)$.

(a) $\det(3A^2B) = \det(3I)\det(A)\det(A)\det(B)$

$= 3^3 \cdot 5 \cdot \det(B)$

$\det(A) = \det(E_3)\det(E_2)\det(E_1)\det(B)$

$\sqrt{5} = (-4)(1)(-1)\det(B)$

$\det(3A^2B) = \boxed{3^3 \cdot 5\left(\frac{\sqrt{5}}{4}\right)} = \frac{135\sqrt{5}}{4}$

(b) $C = A^2 + 2AB - 2BA - 4B^2$

$C = (A-2B)(A+2B)$

$\det(C) = 0$ because $\det(A^T) = 0$ due to combo rule applied to the two duplicate rows.

$0 = \det(A-2B)\det(A+2B)$

$0 = \det(A-2B)(-5)$

$\boxed{\det(A-2B) = 0}$

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5. (inverses and Cramer's rule) Do all three parts.

(a) [20%] Determine all values of x for which A^{-1} exists: $A = \begin{pmatrix} 1 & 2x+1 & 0 \\ 4 & 3 & 0 \\ 5x & -44x & 11x^2 \end{pmatrix}$.

(b) [40%] Apply the adjugate formula for the inverse to find the value of the entry in row 3, column 4 of A^{-1}, given A below. Other methods are not acceptable.

$$A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ -1 & 0 & -1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 2 \end{pmatrix}$$

(c) [40%] Solve for y in $A\mathbf{u} = \mathbf{b}$ by Cramer's rule. Other methods are not acceptable.

$A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 0 & 4 \\ 5 & 6 & 7 \end{pmatrix}$, $\mathbf{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

(a) $\det(A) = 11x^2 \begin{vmatrix} 1 & 2x+1 \\ 4 & 3 \end{vmatrix} = 11x^2(-1-8x)$

A^{-1} exists for $x \neq 0$ and $x \neq -1/8$ (A^{-1} exists $\Leftrightarrow \det(A) \neq 0$)

(b) $\text{entry}(3,4) = \dfrac{\text{cofactor}(A,4,3)}{\det(A)} = \dfrac{(-1)\begin{vmatrix} 1 & 2 & 1 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{vmatrix}}{1} = \dfrac{-1}{1} = \boxed{-1}$

$\det(A) = (-1)(-1)\begin{vmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 2 \end{vmatrix} + (1)(1)\begin{vmatrix} 1 & 2 & 1 \\ -1 & 0 & 0 \\ 1 & 2 & 2 \end{vmatrix}$ cofactor expansion col 3

$= \begin{vmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + 2$ two combos

$= -1 + 2 = 1$

(c) $y = \dfrac{\begin{vmatrix} 1 & 1 & 0 \\ 3 & 0 & 4 \\ 5 & -1 & 7 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 0 \\ 3 & 0 & 4 \\ 5 & 6 & 7 \end{vmatrix}} = \dfrac{\begin{vmatrix} 0 & 4 \\ -1 & 7 \end{vmatrix} - \begin{vmatrix} 3 & 4 \\ 5 & 7 \end{vmatrix}}{\begin{vmatrix} 0 & 4 \\ 6 & 7 \end{vmatrix} - 2\begin{vmatrix} 3 & 4 \\ 5 & 7 \end{vmatrix}} = \dfrac{4-1}{-24-2} = \boxed{-\dfrac{3}{26}}$

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