Name KEY Classtime ______ 2250 Midterm 2 Ver 1 [7:30, S2007]

Applied Differential Equations 2250

Exam date: Tuesday, 13 March, 2007

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%.

1. (Frame sequences and the 3 properties)
Determine a, b such that the system has a unique solution, infinitely many solutions, or no solution:

$$\begin{array}{rcrcrcl} x &+& 2y &+& z &=& a \\ 3x &+& 3by &+& 2z &=& b \\ 4x &+& 8y &+& 3z &=& 2+a \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & a \\ 3 & 3b & 2 & b \\ 4 & 8 & 3 & 2+a \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & a \\ 0 & 3b-6 & -1 & b-3a \\ 4 & 8 & 3 & 2+a \end{array}\right) \text{combo}(1,2,-3)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & a \\ 0 & 3b-6 & -1 & b-3a \\ 0 & 0 & -1 & 2-3a \end{array}\right) \text{combo}(1,3,-4)$$

Case $3b-6=0$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & a \\ 0 & 0 & -1 & 2-3a \\ 0 & 0 & -1 & 2-3a \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & a \\ 0 & 0 & -1 & 2-3a \\ 0 & 0 & 0 & 0 \end{array}\right) \text{combo}(2,3,-1)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & a \\ 0 & 0 & 1 & 3a-2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Two lead variables

$\boxed{\infty\text{-many solutions for } b=2}$

Case $3b-6 \neq 0$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & a \\ 0 & 1 & x & y \\ 0 & 0 & 1 & 3a-2 \end{array}\right)$$

$x = \dfrac{-1}{3b-6}$, $y = \dfrac{b-3a}{3b-6}$

Three leading ones

$\boxed{\text{unique solution for } b \neq 2}$

$\boxed{\begin{array}{l}\text{Never a signal equation} \\ \text{No Sol does not happen}\end{array}}$

Use this page to start your solution. Attach extra pages as needed, then staple.

Name KEY Classtime ______ 2250 Midterm 2 Ver 1 [7:30, S2007]

2. **(vector spaces)** Do all three parts.

(a) [20%] The vector space V is the set of all polynomials $p(x) = x^2(a_0 + a_1 x + a_2 x^5)$. Find a subspace S of V of dimension 2 and display a basis for S. Don't justify anything.

(b) [40%] Let V be the vector space of all column vectors $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and let S be the subset of V given by the equations $x_1 + 2x_3 = 0$, $x_2 x_3 = 0$. Prove that S is not a subspace of V.

(c) [40%] Find a basis of vectors for the subspace of $\mathcal{R}^4$ given by the system of equations

$$\begin{array}{rcrcrcrcl} x_1 & + & 2x_2 & - & 2x_3 & + & x_4 & = & 0, \\ x_1 & + & x_2 & - & 3x_3 & + & x_4 & = & 0, \\ & & 2x_2 & + & 2x_3 & & & = & 0. \end{array}$$

(a) $p = a_0 x^2 + a_1 x^3 + a_2 x^7$. A basis of V is $\frac{\partial P}{\partial a_0} = x^2$, $\frac{\partial P}{\partial a_1} = x^3$, $\frac{\partial P}{\partial a_2} = x^7$. Let $S = \text{span}\{x^2, x^3\}$ with basis $= \{x^2, x^3\}$. Then S is a subspace of V of dimension 2.

(b) $\vec{v}_1 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ are in S. But $\vec{v} = \vec{v}_1 + \vec{v}_2 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$ is not (because $x_2 = x_3 = 1$ violates $x_2 x_3 = 0$). By the Subspace criterion (Thm 1 in E&P), S is not a subspace of V.

(c) $A = \begin{pmatrix} 1 & 2 & -2 & 1 \\ 1 & 1 & -3 & 1 \\ 0 & 2 & 2 & 0 \end{pmatrix}$

$\text{rref}(A) = \begin{pmatrix} 1 & 0 & -4 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

gen scalar sol

$$\begin{cases} x_1 = 4t_1 - t_2 \\ x_2 = -t_1 \\ x_3 = t_1 \\ x_4 = t_2 \end{cases}$$

vector gen sol

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = t_1 \begin{pmatrix} 4 \\ -1 \\ 1 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\boxed{\text{Basis} = \left\{ \begin{pmatrix} 4 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}}$$

Use this page to start your solution. Attach extra pages as needed, then staple.

Name KEY Classtime ______ 2250 Midterm 2 Ver 1 [7:30, S2007]

3. **(independence)** Do **only two** of the three parts.

(a) [50%] Let $\mathbf{u}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{u}_3 = \begin{pmatrix} 1 \\ 1 \\ 3 \\ 3 \end{pmatrix}$, State a test that decides independence or dependence of the list of three vectors [20%]. Apply the test and report the result [30%].

(b) [50%] State the pivot theorem. Then extract from the list below a largest set of independent vectors.

$$\mathbf{a} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}. \ \mathbf{b} = \begin{pmatrix} 2 \\ -2 \\ 0 \\ 2 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 2 \end{pmatrix}, \mathbf{d} = \begin{pmatrix} 3 \\ -1 \\ 0 \\ 5 \end{pmatrix}, \mathbf{e} = \begin{pmatrix} 3 \\ 3 \\ 0 \\ 9 \end{pmatrix},$$

(c) [50%] [If you did (a) and (b) already, then 100% has been attained: skip this one!]
Assume that 3×2 matrix D has rank 2. Prove that there exists an invertible matrix E such that $ED = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$.

(a) Vectors u_1, u_2, u_3 in $\mathbb{R}^4$ are independent $\Leftrightarrow$ rank$(A) = 3$ where $A =$ augmented matrix of u_1, u_2, u_3.

$$A = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & 3 \\ 1 & 1 & 3 \end{pmatrix} \quad \text{rref}(A) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{rank}(A) = 2$$

u_1, u_2, u_3 are <u>dependent</u>

(b) Pivot Theorem. [20%] 1. The pivot columns of a matrix A are independent. 2. The non-pivot columns are linear combinations of the pivot columns of A.

$A = \text{aug}(a, b, c, d, e)$ [30%] $\quad \text{rref}(A) = \begin{pmatrix} 1 & 2 & 0 & 3 & 3 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ $\quad \boxed{\vec{a}, \vec{c}}$

(c) If 3×2 D has rank 2, then rref(D) has two leading ones. The ones appear in columns of the identity in natural order. Then rref$(D) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$. By frame sequence Theory, there are elementary matrices $E_1, \ldots, E_k$ such that

$$\text{rref}(D) = E_k \cdots E_1 D$$

Let $E = E_k \cdots E_1$. Then E is invertible, because it is the product of (elementary) invertible matrices. Finally, the identity for ED holds.

Use this page to start your solution. Attach extra pages as needed, then staple.

Name KEY Classtime ______ 2250 Midterm 2 Ver 1 [7:30, S2007]

4. (determinants and elementary matrices) Do both parts.
(a) [50%] Assume given 3×3 matrices A, B. Suppose $E_5E_4B = E_3E_2E_1A$ and E_1, E_2, E_3, E_4, E_5 are elementary matrices representing respectively a swap, a combination, a multiply by 3, a swap and a multiply by 7. Assume $\det(A) = 5$. Find $\det(5A^2B)$.
(b) [50%] Let A, B and C be 4×4 matrices such that $C + B^2 + BA = A^2 + AB$. Suppose C contains a pair of duplicate rows and $\det(A - B) = -11$. Find the value of $\det(A + B)$.

(a) $\det(E_5)\det(E_4)\det(B) = \det(E_3)\det(E_2)\det(E_1)\det(A)$

$(7)\ (-1)\ \det(B) = (3)\ (1)\ (-1)\ (5)$

$\det(B) = \frac{15}{7}$

$\det(5A^2B) = \det(5I)\det(A)\det(A)\det(B)$

$= 5^3 \cdot 5 \cdot 5 \left(\frac{15}{7}\right)$

$= \boxed{5^5\left(\frac{15}{7}\right)} = 5^6\left(\frac{3}{7}\right)$

(b) $C = A^2 + AB - BA - B^2$

$C = (A-B)(A+B)$

$\det(C) = 0$ because a combo can make a row of zeros

$0 = \det(A-B)\det(A+B)$

$0 = (-11)\det(A+B)$

$\boxed{\det(A+B) = 0}$

Use this page to start your solution. Attach extra pages as needed, then staple.

Name KEY Classtime ______ 2250 Midterm 2 Ver 1 [7:30, S2007]

5. (inverses and Cramer's rule) Do all three parts.

(a) [20%] Determine all values of x for which A^{-1} fails to exist: $A = \begin{pmatrix} 1 & 2x-1 & 0 \\ 2 & 3 & 0 \\ 5x & -44x & 64x^2 \end{pmatrix}$.

(b) [40%] Apply the adjugate formula for the inverse to find the value of the entry in row 2, column 3 of A^{-1}, given A below. Other methods are not acceptable.

$$A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ -1 & 0 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 2 \end{pmatrix}$$

(c) [40%] Solve for z in $A\mathbf{u} = \mathbf{b}$ by Cramer's rule. Other methods are not acceptable.

$A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 0 & 4 \\ 5 & 6 & 8 \end{pmatrix}$, $\mathbf{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

(a) A^{-1} fails to exist $\Leftrightarrow$ $\det(A) = 0$.

$\det(A) = (+1)(64x^2)\begin{vmatrix} 1 & 2x-1 \\ 2 & 3 \end{vmatrix} = 6x^2(5-4x)$

$\boxed{\begin{array}{l} x = 0 \\ x = 5/4 \end{array}}$

(b) $\text{entry}(A^{-1}, 2, 3) = \dfrac{\text{Cofactor}(A,3,2)}{\det(A)} = \dfrac{1}{1} = \boxed{1}$

$\text{cofactor} = (-1)\begin{vmatrix} 1 & 0 & 1 \\ -1 & -1 & 1 \\ 1 & 0 & 2 \end{vmatrix} = (-1)\left(\begin{vmatrix} -1 & 1 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} -1 & -1 \\ 1 & 0 \end{vmatrix}\right) = (-1)(-2+1) = 1$

along row 1

$\det(A) = (-1)(-1)\begin{vmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 2 \end{vmatrix} + (1)(1)\begin{vmatrix} 1 & 2 & 1 \\ -1 & 0 & 1 \\ 1 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 2 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{vmatrix} = -1+2 = 1$

cof. exp. along col 3 ← exp. cof. row 2 2 combos 2 combos

(c) $z = \dfrac{\begin{vmatrix} 1 & 2 & 1 \\ 3 & 0 & 0 \\ 5 & 6 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 0 \\ 3 & 0 & 4 \\ 5 & 6 & 8 \end{vmatrix}} = \dfrac{(-3)\begin{vmatrix} 2 & 1 \\ 6 & -1 \end{vmatrix}}{\begin{vmatrix} 0 & 4 \\ 6 & 8 \end{vmatrix} - 2\begin{vmatrix} 3 & 4 \\ 5 & 8 \end{vmatrix}} = \dfrac{(-3)(-8)}{-24-8} = \boxed{\dfrac{-3}{4}}$

exp. cof. row 1

Use this page to start your solution. Attach extra pages as needed, then staple.