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# Differential Equations and Linear Algebra 2250 [10:45]

**Midterm Exam 1**

**Tuesday, 13 February 2007**

**Instructions**: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%. Unevaluated integrals will receive partial credit.

**1. (Quadrature Equation)**

Solve for the general solution $y(x)$ in the equation $y' = e^x \ln(1+e^x) + \cos^3 x + \dfrac{x^2}{4+x^2}$.

$F_1 = e^x \ln(1+e^x)$

$F_2 = \cos^3 x$

$= \cos x\,(1 - \sin^2 x)$

$F_3 = \dfrac{x^2}{4+x^2}$

$= 1 + \dfrac{-4}{x^2+4}$

$\int F_1 dx = \int \ln u \, du$ $\qquad u = 1+e^x$

$= u \ln u - u$

$= (1+e^x)\ln(1+e^x) - (1+e^x)$

$\int F_2 dx = \sin x - \dfrac{\sin^3 x}{3}$

$\int F_3 dx = x + \int \dfrac{-4\,dx}{x^2+4}$

$= x - 2\tan^{-1}(x/2)$

$y = \int (F_1 + F_2 + F_3)\, dx$

$= (1+e^x)\ln(1+e^x) - (1+e^x) +$

$\sin x - \frac{1}{3}\sin^3 x +$

$x - 2\tan^{-1}\left(\frac{x}{2}\right) + c$

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**2. (Classification of Equations)**
The problem $y' = f(x,y)$ is defined to be **separable** provided $f(x,y) = F(x)G(y)$ for some functions $F$ and $G$.

(a) [40%] Check ($\boxed{X}$) the problems that can be put into separable form, but don't supply any details.

| | | | |
|---|---|---|---|
| $\square$ | $y' = y(2xy+1) + (x+1)y$ <br> $= 2xy^2 + 2y + xy$ | $\boxtimes$ | $e^y y' = y^2 \sin x + \sin x$ <br> $y' = e^{-y}(y^2+1)\sin x$ |
| $\boxtimes$ | $y' = e^{3x+2y}$ <br> $= e^{3x}e^{2y}$ | $\boxtimes$ | $y' + e^2 y = e^3$ <br> $y' = e^3 - e^2 y$ |

(b) [25%] State a test which can verify that an equation $y' = f(x,y)$ is linear but not quadrature. Linear: $\frac{\partial f}{\partial y}$ indep of $y$　Quadr: $\frac{\partial f}{\partial y} = 0$

(c) [35%] Use the separable equation test to show that $y' = e^x e^y + x$ is not separable.

$$f(x,y) = e^{x+y} + x$$
$$f(0,0) = e^0 + 0$$
$$= 1$$

$$F(x) = \frac{f(x,0)}{f(0,0)} = e^x + x \qquad G(y) = f(0,y) = e^y$$

$$F(x)G(y) = (e^x + x)e^y$$
$$= e^{x+y} + xe^y$$
$$\neq e^{x+y} + x$$

By the Test, $y' = f(x,y)$ is <u>not separable</u>.

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**3. (Solve a Separable Equation)**

Given $y^2 y' = \left(\frac{\cos x}{\sin^2 x} + \left(\frac{x+1}{2+x}\right)^2\right)(y+1)(y+5)$.

(a) Find all equilibrium solutions.

(b) Find the non-equilibrium solution in implicit form.

To save time, **do not solve** for $y$ explicitly.

(a) $y = -1$ and $y = -5$, found by substitution of $y = c$ into the DE.

(b) $$\frac{y^2 y'}{(y+1)(y+5)} = \sin^{-2}(x)\cos(x) + \left(1 + \frac{-1}{x+2}\right)^2$$

$$F(x) = (\sin x)^{-2}\cos(x) + 1 - \frac{2}{x+2} + \frac{1}{(x+2)^2}$$

$$\int F\,dx = -(\sin x)^{-1} + x - 2\ln|2+x| - \frac{1}{x+2}$$
$$= -\csc x + x - 2\ln|2+x| - \frac{1}{x+2}$$

$$\frac{y^2}{(y+1)(y+5)} = 1 + \frac{-6y-5}{(y+1)(y+5)}$$
$$= 1 + \frac{1/4}{y+1} + \frac{\frac{25}{-4}}{y+5}$$

$$y^2+6y+5 \overline{\big)\, y^2} \quad 1$$
$$\underline{y^2+6y+5}$$
$$-6y-5$$

$$\int \frac{y^2 y'}{(y+1)(y+5)} = y + \frac{1}{4}\ln|y+1| - \frac{25}{4}\ln|y+5|$$

answer:

$$y + \frac{1}{4}\ln|y+1| - \frac{25}{4}\ln|y+5| = -\csc x + x - 2\ln|2+x| - \frac{1}{x+2} + C$$

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**4. (Linear Equations)**

(a) [60%] Solve $x'(t) = -16 + \dfrac{1}{2t+3}x(t)$, $x(0) = -48$. Show all integrating factor steps.

(b) [20%] Solve the homogeneous equation $\dfrac{dy}{dx} = -x^3 y$. The answer contains symbol $c$.

(c) [20%] Solve $y' = 2y + 3$ by using the superposition principle $y = y_h + y_p$.

(a) $x' + \dfrac{-1}{2t+3}x = -16$

$\dfrac{(Wx)'}{W} = -16$

$W = e^{\int \frac{-dt}{2t+3}}$

$= e^{-\frac{1}{2}\ln|2t+3|}$

$= (2t+3)^{-1/2}$ for $t$ near 0

$(Wx)' = -16W$

$Wx = -16\int (2t+3)^{-1/2}dt$

$= -\dfrac{16}{2}\dfrac{(2t+3)^{1/2}}{1/2} + c$

$x = \dfrac{c}{W} - 16\dfrac{(2t+3)^{1/2}}{W}$

$x = c(2t+3)^{1/2} - 16(2t+3)$

$x(0) = -48 \iff c = 0$

$\boxed{x = -16(2t+3)} \boxed{= -32t - 48}$

(b) $\dfrac{(Wy)'}{W} = 0$ $\quad W = e^{\int x^3 dx}$

$Wy = c \implies \boxed{y = ce^{-x^4/4}}$

(c) $y_p = -3/2$ is an equil. sol.

$y' = 2y$ has growth-decay sol $y_h = ce^{2x}$

$y = y_h + y_p$

$\boxed{y = ce^{2x} - 3/2}$

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**5. (Stability)**

(a) [50%] Draw a phase line diagram for the differential equation

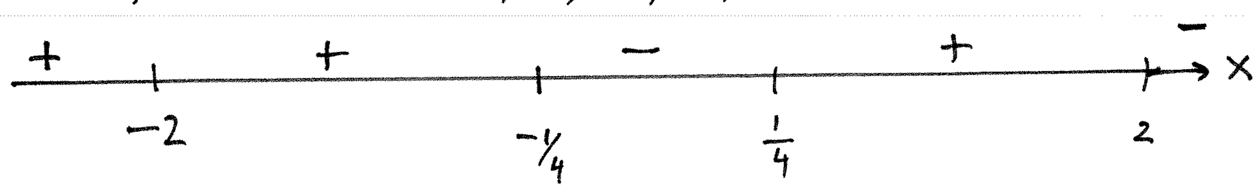
$$\frac{dx}{dt} = e^{-x}\left(1 - \sqrt[3]{|4x|}\right)^2 (2+x)(4-x^2)(16x^2-1)^3.$$

Expected in the diagram are equilibrium points and signs of $x'$.

(b) [50%] Draw a phase diagram using the phase line diagram of (a). Add these labels as appropriate: funnel, spout, node, source, sink, stable, unstable. Show at least 8 threaded curves. A direction field is not expected nor required.

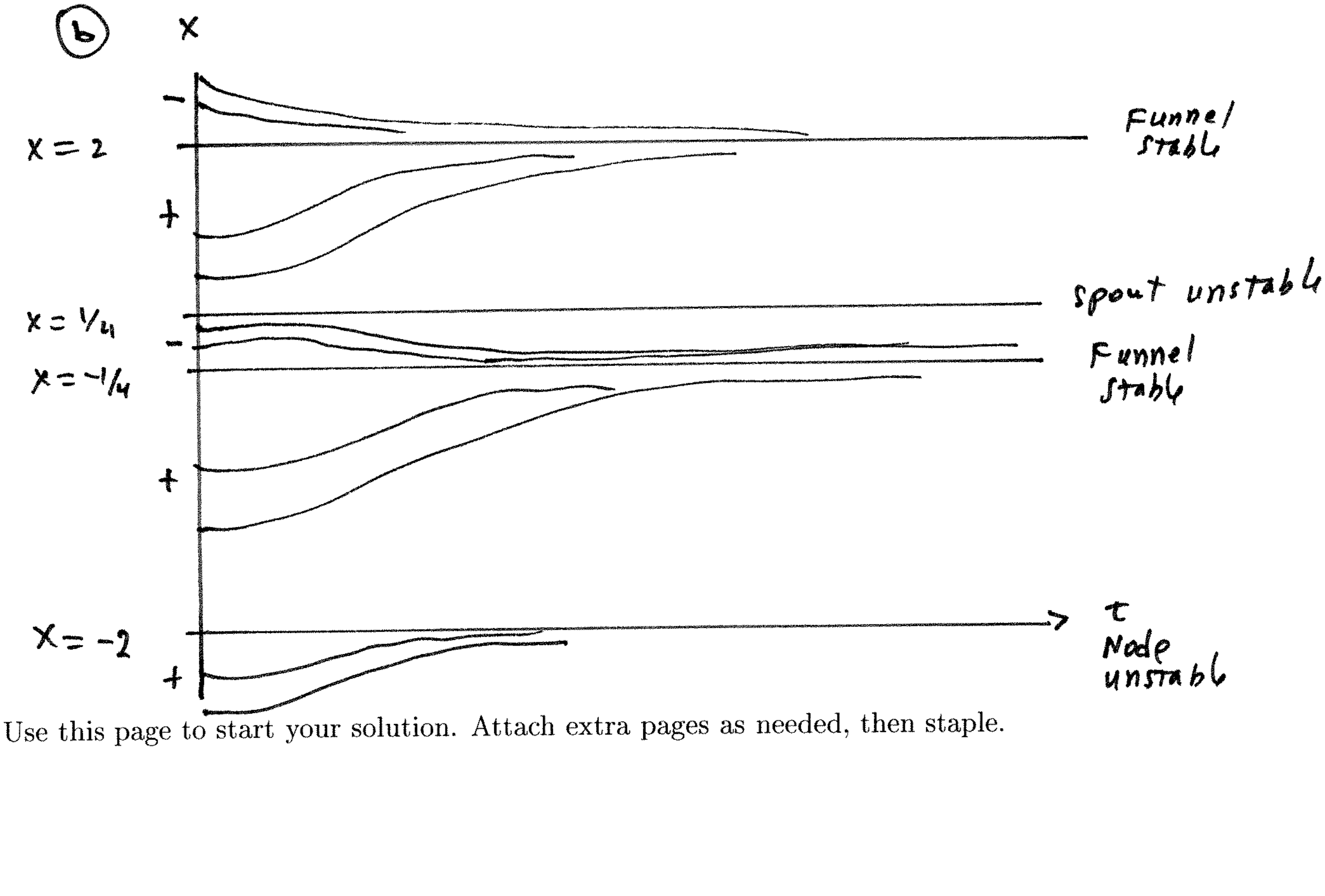
(a) $f(x) = e^{-x}\left(1 - |4x|^{1/3}\right)^2 (2+x)^2 (2-x)(4x-1)^3(4x+1)^3$

$f(x) = 0 \Leftrightarrow x = 2, -2, 1/4, -1/4$




Substitute $x = -3, -1, 0, 1, 3$ to find out the signs of $f$.

(b)

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