Name. KEY Time of your class ______

Differential Equations and Linear Algebra 2250 [7:30]

Midterm Exam 1

Tuesday, 13 February 2007

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%. Unevaluated integrals will receive partial credit.

1. **(Quadrature Equation)**

   Solve for the general solution $y(x)$ in the equation $y' = e^{-x}\ln(1+e^{-x}) + 2\sin x\cos x + \dfrac{x^2}{2+x}$.

$F_1 = e^{-x}\ln(1+e^{-x})$

$F_2 = 2\sin x\cos x$

$F_3 = \dfrac{x^2}{2+x}$

$$\begin{array}{r|l} & x-2 \\ \hline 2+x & x^2 \\ & 2x+x^2 \\ \hline & -2x \\ & -2x-4 \\ \hline & 4 \end{array}$$

$\int F_1 dx = \int -\ln u\, du \qquad u = 1+e^{-x}$

$= -(u\ln u - u)$

$= -(1+e^{-x})\ln(1+e^{-x}) + (1+e^{-x})$

$\int F_2 dx = \int 2u\, du \qquad u = \sin x$

$= u^2$

$= \sin^2(x)$

$\int F_3 dx = \int \left(x-2+\dfrac{4}{2+x}\right)dx$ by division algorithm

$= \dfrac{x^2}{2} - 2x + 4\ln|2+x|$

$y = \int (F_1+F_2+F_3)dx$

$= -(1+e^{-x})\ln(1+e^{-x}) + (1+e^{-x})$

$+ \sin^2(x)$

$+ \dfrac{x^2}{2} - 2x + 4\ln|2+x| + C$

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Name. KEY　　　　Time of your class ______　　　2250 [7:30]

2. **(Classification of Equations)**
The problem $y' = f(x,y)$ is defined to be **separable** provided $f(x,y) = F(x)G(y)$ for some functions F and G.

(a) [40%] Check ($\boxed{X}$) the problems that can be put into separable form, but don't supply any details.

[X]	$y' = y(2xy+1)+(x-1)y$ $= 2xy^2 + xy$	[ ]	$yy' = xy^2 + x^2$
[ ]	$y' = xe^{2y} + ye^{x}$	[X]	$y' + y = e^{\pi}$ $y' = e^{\pi} - y$

(b) [25%] State a test which can verify that an equation $y' = f(x,y)$ is both quadrature and linear. Quadr: $\frac{\partial f}{\partial y} = 0$　　Linear: $\frac{\partial f}{\partial y}$ independent of y

(c) [35%] Use the separable equation test to show that $y' = e^y + e^x$ is not separable.

$$f(x,y) = e^{y} + e^{x}$$
$$f(0,0) = e^{0} + e^{0} = 2$$
$$F(x) = \frac{f(x,0)}{f(0,0)} = \frac{1+e^x}{2} \qquad G(y) = f(0,y) = 1+e^y$$
$$F(x)G(y) = \left(\frac{1+e^x}{2}\right)(1+e^y) = \frac{1}{2}\left(1 + e^x + e^y + e^{x+y}\right) \neq e^x + e^y = f$$

By the test, $y' = f(x,y)$ is <u>not separable</u>.

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Name. KEY Time of your class ______ 2250 [7:30]

3. (Solve a Separable Equation)

Given $y^3y' = \left(\csc x \cot x + \left(\frac{x+1}{3+x}\right)^2\right)(5-y)(y-1)$.

(a) Find all equilibrium solutions.

(b) Find the non-equilibrium solution in implicit form.

To save time, **do not solve** for y explicitly.

(a) $y=1$ and $y=5$, found by setting $y=c$ in the DE.

(b) $\frac{y^3y'}{(5-y)(y-1)} = \csc x \cot x + \left(\frac{x+1}{3+x}\right)^2$

$$3+x \overline{)\, x+1 \,}^{\;1}$$
$$\frac{x+3}{-2}$$

$$F(x) = \csc x \cot x + \left(\frac{x+1}{3+x}\right)^2$$
$$= (\sin x)^{-2}\cos x + \left(1 + \frac{-2}{3+x}\right)^2$$
$$= (\sin x)^{-2}\cos x + 1 - \frac{4}{3+x} + \frac{4}{(3+x)^2}$$

$$\int F\,dx = -(\sin x)^{-1} + x - 4\cdot\ln|3+x| - \frac{4}{3+x} \qquad [(\sin x)^{-1} = \csc x]$$

$$\frac{y^3}{(5-y)(y-1)} = -y - 6 + \frac{A}{y-1} + \frac{B}{y-5}$$

by long division

$$y^2-6y+5 \overline{)\, -y^3 \,}^{\;-y-6}$$
$$\vdots$$
$$-31y+30$$

$$\int \frac{y'y^3}{(5-y)(y-1)} = -\frac{y^2}{2} - 6y + A\ln|y-1| + B\ln|y-5|$$

by partial fractions

$A = \frac{1}{4} \quad B = \frac{-125}{4}$

answer:

$$-\frac{y^2}{2} - 6y + \frac{1}{4}\ln|y-1| - \frac{125}{4}\ln|y-5| = \csc x + x - 4\ln|3+x| - \frac{4}{3+x} + c$$

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Name. KEY

Time of your class ______ 2250 [7:30]

4. **(Linear Equations)**

(a) [60%] Solve $x'(t) = -16 + \frac{1}{2t+2}x(t)$, $x(0) = -32$. Show all integrating factor steps.

(b) [20%] Solve the homogeneous equation $\frac{dy}{dx} = -x^2 y$. The answer contains symbol c.

(c) [20%] Solve $y' + 3y = 13$ using the superposition principle $y = y_h + y_p$.

ⓐ $x' + \frac{-1}{2t+2}x = -16$

$W = e^{\int \frac{-dt}{2t+2}} = e^{-\frac{1}{2}\ln|1+t|} = (1+t)^{-1/2}$ valid near $t=0$

$\frac{(Wx)'}{W} = -16$

$(Wx)' = -16W$

$Wx = -16\int (1+t)^{-1/2}dt + c$

$= -16\frac{(1+t)^{1/2}}{1/2} + c$

$x = -32(1+t) + c(1+t)^{1/2}$

$x(0) = -32 \Leftrightarrow c = 0$

$\boxed{x = -32(1+t)}$

ⓑ $y' + x^2 y = 0$

$\frac{(Wy)'}{W} = 0$

$W = e^{\int x^2 dx} = e^{x^3/3}$

$y = c/W$

$\boxed{y = ce^{-x^3/3}}$

ⓒ $y_p = 13/3$, an equil. sol.

$y' + 3y = 0$ has Growth-Decay sol $y_h = ce^{-3x}$

$y = y_h + y_p$

$\boxed{y = ce^{-3x} + \frac{13}{3}}$

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Name. KEY   **Time of your class** ______   2250 [7:30]

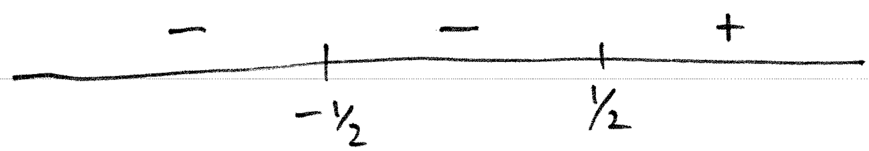
5. (Stability)
(a) [50%] Draw a phase line diagram for the differential equation

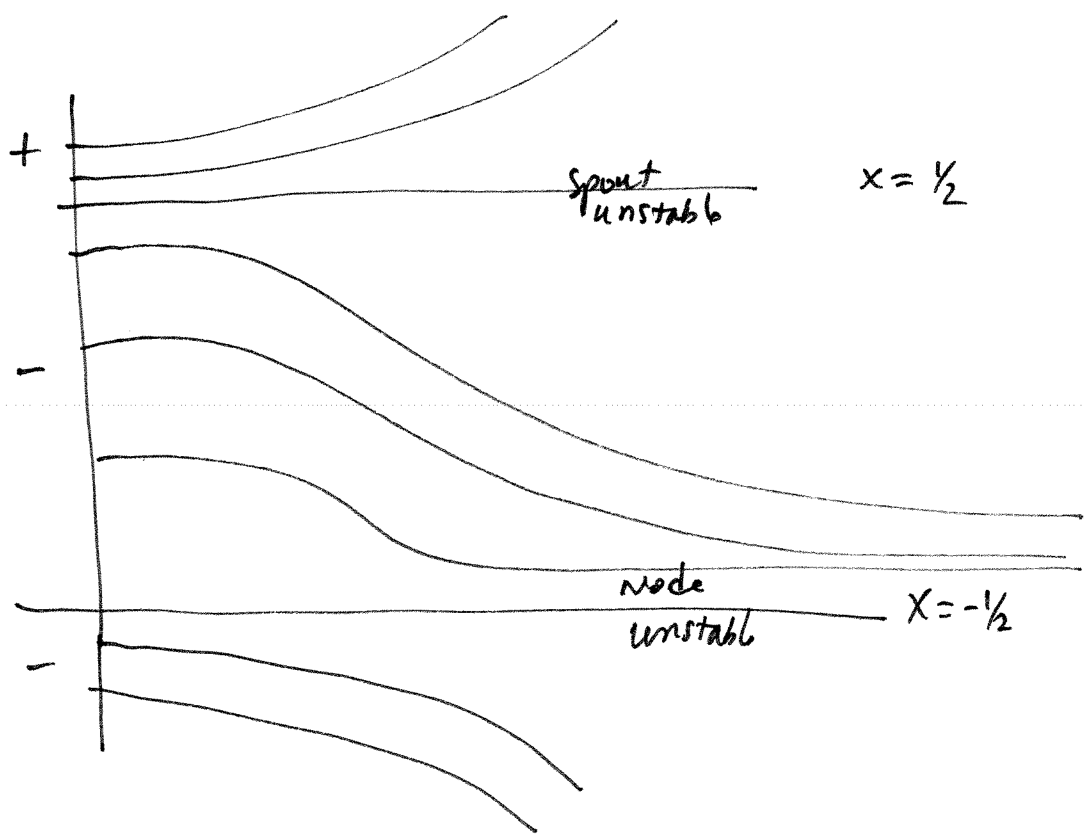
$$dx/dt = e^x\left(1-\sqrt[3]{|2x|}\right)^3(1+2x)(1-4x^2)(36x^2-9)^3.$$

$x' = f(x)$

Expected in the diagram are equilibrium points and signs of x' (or flow direction markers < and >).
(b) [50%] Draw a phase diagram using the phase line diagram of (a). Add these labels as appropriate: funnel, spout, node, source, sink, stable, unstable. Show at least 8 threaded curves. A direction field is not expected nor required.

$f(-1) = -$ $f(0) = -$ $f(1) = +$





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