

**Math 2250 Extra Credit Problems**  
**Chapter 4**  
**March 2007**

**Due date:** See the internet due date for 6.1, which is the due date for these problems. Records are locked on that date and only corrected, never appended.

**Submitted work.** Please submit one stapled package per problem. Kindly label problems **Extra Credit**. Label each problem with its corresponding problem number. You may attach this printed sheet to simplify your work.

**Problem ExL3.1. (maple lab 3)**

You may submit this problem only for score increases on maple lab 3.

Solve symbolically by chapter 1 methods the initial value problem  $y' = 2xy^2$ ,  $y(0) = 1$ . Do an answer check in maple or by hand. Answer:  $y = 1/(1 - x^2)$ . This problem has no numerical work!

**Problem ExL3.2. (maple lab 3)**

You may submit this problem only for score increases on maple lab 3. This problem counts as three (3) problems.

Solve  $y' = 2xy^2$ ,  $y(0) = 1$  numerically for the value of  $y(0.5)$  using (1) Euler's method, (2) Heun's method and (3) the RK4 method. Use step size  $h = 0.1$ . Include computer code and a print of the data. Report the answers in a table for  $x$ -values 0, 0.1, 0.2, 0.3, 0.4, 0.5.

**Problem ExL4.1. (maple lab 4)**

You may submit this problem only for score increases on maple lab 4.

Solve symbolically by chapter 1 methods the initial value problem  $y' = e^{-y}$ ,  $y(0) = 0$ . Do an answer check in maple or by hand. Answer:  $y = \ln(1 + x)$ . This problem has no numerical work!

**Problem ExL4.2. (maple lab 4)**

You may submit this problem only for score increases on maple lab 4. This problem counts as three (3) problems.

Solve  $y' = e^{-y}$ ,  $y(0) = 0$  numerically for the value of  $y(1.0)$  using (1) Euler's method, (2) Heun's method and (3) the RK4 method. Use step size  $h = 0.001$ . Include a computer code appendix in the report, but do not print the data. Report the answers in a table for  $x$ -values 0, 0.2, 0.4, 0.6, 0.8, 1.0. Include the percentage error  $E = 100|\ln(2) - y(1.0)|/|\ln(2)|$  in your report, one error report for each of the three methods.

**Problem Ex4.1-20. (Independence)**

Test independence or dependence:  $\begin{pmatrix} 5 \\ 5 \\ 4 \end{pmatrix}$ ,  $\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$ ,  $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ .

**Problem Ex4.1-30. (Kernel theorem)**

Verify from the kernel theorem (Theorem 2, 4.2) that the set of all vectors in  $\mathcal{R}^3$  such that  $2x - y = 3z$  is a subspace of  $\mathcal{R}^3$ .

**Problem Ex4.1-32. (Subspace criterion)**

Apply the subspace criterion (Theorem 1, 4.2) to verify that the set of all vectors in  $\mathcal{R}^3$  such that  $2x - y = 3z$ ,  $xy = 0$  is **not** a subspace of  $\mathcal{R}^3$ .

**Problem Ex4.2-4. (Subspace criterion)**

Apply the subspace criterion (Theorem 1, 4.2) to verify that the set of all vectors in  $\mathcal{R}^3$  satisfying  $|x| = y + z$  fails to be a subspace of  $\mathcal{R}^3$ .

**Problem Ex4.2-28. (Kernel theorem)**

Let  $S$  be the subset of  $\mathcal{R}^4$  defined by the equations

$$\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 4 & 0 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Use the kernel theorem (Theorem 2, 4.2) to prove that  $S$  is a subspace of  $\mathcal{R}^4$ .

**Problem Ex4.3-18. (Dependence and frame sequences)**

Give the vectors below, display a frame sequence from the augmented matrix  $C$  of the vectors to final frame  $\mathbf{rref}(C)$ . Use the sequence to decide if the vectors are independent or dependent. If dependent, then report all possible dependency relations  $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$ .

$$\mathbf{v}_1 = \begin{pmatrix} 3 \\ 9 \\ 0 \\ 5 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -2 \\ 4 \\ -10 \\ 10 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 4 \\ 7 \\ 5 \\ 0 \end{pmatrix}.$$

**Problem Ex4.3-24. (Independence in abstract vector spaces)**

Let  $V$  be an abstract vector space whose packages of data items  $\mathbf{v}$  have unknown details. You are expected to use only definitions and the toolkit of 8 properties in the details of this problem.

Assume given  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  independent vectors in  $V$ . Define  $\mathbf{u}_1 = \mathbf{v}_1 + 2\mathbf{v}_2$ ,  $\mathbf{u}_2 = \mathbf{v}_3 + \mathbf{u}_1$ ,  $\mathbf{u}_3 = \mathbf{v}_1 + \mathbf{u}_2$ . Prove that  $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$  are independent.

**Problem Ex4.4-6. (Basis)**

Find a basis for  $\mathcal{R}^3$  which includes two independent vectors  $\mathbf{v}_1, \mathbf{v}_2$  which are in the plane  $2x - 3y + 5z = 0$  and a vector  $\mathbf{v}_3$  which is perpendicular to both  $\mathbf{v}_1$  and  $\mathbf{v}_2$ . Hint: review the cross product, a topic from calculus.

**Problem Ex4.4-24. (Basis for  $A\mathbf{x} = \mathbf{0}$ )**

Display a frame sequence starting with  $A$  having final frame  $\mathbf{rref}(A)$ . Use this sequence to find the scalar general solution of  $A\mathbf{x} = \mathbf{0}$  and then the vector general solution of  $A\mathbf{x} = \mathbf{0}$ . Finally, report a basis for the solution space of  $A\mathbf{x} = \mathbf{0}$ .

$$A = \begin{pmatrix} 1 & -4 & -3 & -7 \\ 2 & -1 & 1 & 7 \\ 1 & 2 & 3 & 11 \\ 2 & 6 & 4 & 8 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

**Problem Ex4.5-6. (Row and columns spaces)**

Find a basis for the row space and the column space of  $A$ , but the bases reported must be rows of  $A$  and columns of  $A$ , respectively.

$$A = \begin{pmatrix} 1 & -4 & -3 & -7 \\ 2 & -1 & 1 & 7 \\ 2 & 6 & 4 & 8 \end{pmatrix}$$

**Problem Ex4.5-24. (Redundant columns)**

Use the pivot theorem (Algorithm 2, 4.5) to find the non-pivot columns of  $A$  (called redundant columns).

$$A = \begin{pmatrix} 1 & -4 & -3 & -7 & 4 & 3 \\ 2 & -1 & 1 & 7 & 2 & 0 \\ 1 & 2 & 3 & 11 & 0 & 0 \\ 2 & 6 & 4 & 8 & 0 & 1 \end{pmatrix}$$

**Problem Ex4.5-28. (Rank and the three properties)**

Suppose  $A$  is a  $5 \times 4$  matrix and  $A\mathbf{x} = \mathbf{0}$  has a basis of size 3. What are the possible forms of  $\mathbf{rref}(A)$ ?

**Problem Ex4.6-2. (Orthogonality)**

Let  $A$  be a  $3 \times 3$  matrix such that  $AA^T = I$ . Prove that the columns  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  of  $A$  satisfy the orthogonality relations

$$|\mathbf{v}_1| = |\mathbf{v}_2| = |\mathbf{v}_3| = 1, \quad \mathbf{v}_1 \cdot \mathbf{v}_2 = \mathbf{v}_2 \cdot \mathbf{v}_3 = \mathbf{v}_3 \cdot \mathbf{v}_1 = 0.$$

**Problem Ex4.7-10. (Subspaces of function spaces)**

Let  $V$  be the function space of all polynomials of degree less than 5. Define  $S$  to be the subset of  $V$  consisting of all polynomials  $p(x)$  in  $V$  such that  $p(0) = p(1)$  and  $\int_{-1}^1 p(x)dx = p(2)$ . Prove that  $S$  is a subspace of  $V$ .

**Problem Ex4.7-20. (Partial fractions and independence)**

Assume the polynomials  $1, x, \dots, x^n$  are *independent*. They are a basis for a vector space  $V$ . Use this fact explicitly in the details for determining the constants  $A, B, C, D$  in the partial fraction expansion

$$\frac{x^2 - x + 1}{(x - 1)(x + 1)^2(x - 3)} = \frac{A}{x - 1} + \frac{B}{x + 1} + \frac{C}{(x + 1)^2} + \frac{D}{x - 3}.$$

**Problem Ex4.7-26. (Solution space basis for a DE)**

Find the general solution of  $3y'' + 5y' = 0$ , containing symbols  $c_1, c_2$  for the arbitrary constants. Take partial derivatives  $\partial y/\partial c_1, \partial y/\partial c_2$  to identify two functions of  $x$ . Prove that these functions are independent and hence find a basis for the solution space of the differential equation. Suggestion: View  $3y'' + 5y' = 0$  as two equations  $3v' + 5v = 0$  and  $y' = v$ . Then use elementary first order differential equation methods.

**End of extra credit problems chapter 4.**