KEY

Differential Equations and Linear Algebra

2250-1 7:30am 27 April 2007

Instructions. The time allowed is 120 minutes. The examination consists of six problems, one for each of chapters 3, 4, 5, 6, 7, 10, each problem with multiple parts. A chapter represents 20 minutes on the final exam.

The final exam counts as two midterm exams. For example, if exam scores earned were 90, 91, 92 and the final exam score is 89, then the exam average for the course is

$$\frac{90+91+92+89+89}{5}.$$

Each problem represents several textbook problems numbered (a), (b), (c), $\cdots$. Choose the problems to be graded by check-mark $\boxed{\text{X}}$; the credits should add to 100. Each chapter (Ch3, Ch4, Ch5, Ch6, ch7, Ch10) adds at most 100 towards the maximum final exam score of 600. The final exam score is reported as a percentage 0 to 100, which is the sum of the scores earned on six chapters divided by 600 to make a fraction, then converted to a percentage.

Calculators, books, notes and computers are not allowed.

Details count. Solutions requiring details are considered incomplete. Less than full credit is earned, in this case, for an answer only.

Answer checks are not expected or required. First drafts are expected, not complete presentations.

Please submit **exactly six** separately stapled packages of problems, one package per chapter.

Please discard this page or keep it for your records.

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Ch3. (Linear Systems and Matrices)

☐ [25%] Ch3(a): Let B be the invertible matrix given below, where $\boxed{?}$ means the value of the entry does not affect the answer to this problem. The second matrix C is the adjugate (or adjoint) of B. Find the value of $\det(2B^{-1})$.

$$B = \begin{pmatrix} ? & ? & ? & 0 \\ 0 & -1 & 2 & 0 \\ 1 & 1 & 0 & 0 \\ ? & ? & ? & -3 \end{pmatrix}, \quad C = \begin{pmatrix} 6 & 6 & 12 & 0 \\ -6 & -6 & 6 & 0 \\ -3 & 6 & 3 & 0 \\ 2 & 2 & 4 & -6 \end{pmatrix}$$

☐ [25%] Ch3(b): State the three possibilities for a linear system $A\mathbf{x} = \mathbf{b}$ [5%]. Determine which values of k correspond to these three possibilities, for the system $A\mathbf{x} = \mathbf{b}$ given in the display below [20%].

$$A = \begin{pmatrix} 2 & 3 & -k \\ 2 & k & -2 \\ 0 & 0 & k-1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

☐ [25%] Ch3(c): Assume matrix A is $n \times n$. State three different linear algebra theorems, each with the conclusion that $A\mathbf{x} = \mathbf{b}$ has a unique solution. No proofs, please!

☐ [25%] Ch3(d): Let A be an $n \times n$ triangular matrix with diagonal entries $2j\sin^2(2\pi j/n)$ for $j = 1$ to n. Prove that $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions $\mathbf{x}$.

If you solved (a), (b), (c) and (d), then go on to Ch4. Otherwise, try (e) and (f). Maximum credit is 100%. Parts (e), (f) replace one of (a), (b), (c) or (d).

☐ [15%] Ch3(e): Find the value of x_1 by Cramer's Rule in the system $C\mathbf{x} = \mathbf{b}$, given C and $\mathbf{b}$ below. Evaluate determinants by any hybrid method (triangular, swap, combo, multiply, cofactor). The use of 2×2 Sarrus' rule is allowed. The 3×3 Sarrus' rule is **disallowed**.

$$C = \begin{pmatrix} 1 & 1 & -1 & 0 \\ 1 & 3 & -2 & 1 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 2 & 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 0 \\ -1 \\ -1 \\ 1 \end{pmatrix}$$

☐ [10%] Ch3(f): Give an example of a matrix M with 4 rows and 2 columns such that $M\mathbf{x} = \mathbf{0}$ has a unique solution $\mathbf{x}$.

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Staple this page to the top of all Ch3 work. Submit one package per chapter.

Ch 3 (a) $\det(B)I = BC$ by the adjugate formula $A^{-1} = \frac{\mathrm{adj}(A)}{\det(A)}$.

$\det(B) = \mathrm{row}(B,3)\,\mathrm{col}(C,3) = \boxed{18}$ $\quad \det(2B^{-1}) = \frac{\det(2I)}{\det(B)} = \boxed{\frac{8}{9}}$

Ch 3 (b) $\begin{pmatrix} 2 & 3 & -k & 1 \\ 2 & k & -2 & 2 \\ 0 & 0 & k-1 & 0 \end{pmatrix}$ Frame 1

$\begin{pmatrix} 2 & 3 & -k & 1 \\ 0 & k-3 & k+2 & 1 \\ 0 & 0 & k-1 & 0 \end{pmatrix}$ Frame 2

$k=1$ $\quad \infty$ – many solutions

$\begin{pmatrix} 2 & 3 & -1 & 1 \\ 0 & -2 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ Frame 2

$k=3$ $\quad$ No solution

$\begin{pmatrix} 2 & 3 & -3 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 0 \end{pmatrix}$ Frame 2

$\begin{pmatrix} 2 & 3 & -3 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ Frame 5

signal eq

otherwise $(k-1)(k-3) \neq 0 \Rightarrow$ unique sol.

ch 3 (c) Matrix A $n \times n$. $A\vec{x} = \vec{b}$.

$\mathrm{rref}(A) = I \Rightarrow$ unique sol

$\mathrm{rank}(A) = n \Rightarrow$ unique sol

$\mathrm{nullity}(A) = 0 \Rightarrow$ unique sol

other theorems are possible:

$\det(A) \neq 0 \Rightarrow$ unique sol

ch 3 (d) For $j = n$, the entry on the diagonal is $2n \sin^2(2\pi) = 0$. Then A has a row of zeros. So $\mathrm{rank}(A) < n$ and there are ∞ – many sols for $A\vec{x} = \vec{0}$.

ch 3 (e) By cofactor expansion along row 4, $\det(C) = (-2)\begin{vmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 0 & 0 \end{vmatrix} + (1)\begin{vmatrix} 1 & 1 & -1 \\ 1 & 3 & -2 \\ 0 & 0 & 4 \end{vmatrix} = (4)\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = 8$. By cofactor expansion along row 1,

$$\Delta_1 = \begin{vmatrix} 0 & 1 & -1 & 0 \\ -1 & 3 & -2 & 1 \\ -1 & 0 & 4 & 0 \\ 1 & 0 & 2 & 1 \end{vmatrix} = (-1)\begin{vmatrix} -1 & -2 & 1 \\ -1 & 4 & 0 \\ 1 & 2 & 1 \end{vmatrix} + (-1)\begin{vmatrix} -1 & 3 & 1 \\ -1 & 0 & 0 \\ 1 & 0 & 1 \end{vmatrix} = (-1)\begin{vmatrix} -1 & -2 & 1 \\ 0 & 6 & -1 \\ 0 & 0 & 2 \end{vmatrix} + (-1)(-3)(-1)$$

$= 9$

$x_1 = \frac{\Delta_1}{\Delta} = \boxed{\frac{9}{8}}$

ch 3 (f) $M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$

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Ch4. (Vector Spaces)

☐ [25%] Ch4(a): State (1) a rank test and (2) a determinant test to detect the independence or dependence of fixed vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ [10%]. Apply one of the tests to the vectors below [10%]. Report **independent** or **dependent** [5%].

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -1 \\ 0 \\ 3 \\ 1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} -1 \\ 2 \\ 7 \\ 2 \end{pmatrix}.$$

☐ [25%] Ch4(b): Let $\mathbf{v}_1$, $\mathbf{v}_2$ be a basis for a subspace S of vector space $V = \mathcal{R}^n$. Let $\mathbf{w}_1$, $\mathbf{w}_2$ be any other proposed basis of S. Define B to be the augmented matrix of $\mathbf{v}_1$, $\mathbf{v}_2$ and define C to be the augmented matrix of $\mathbf{w}_1$, $\mathbf{w}_2$. Does $\mathbf{rref}(B) = \mathbf{rref}(C)$ imply the bases are equivalent [10%]? Explain [15%].

☐ [50%] Ch4(c): Define the 4×4 matrix A by the display below. Find a basis of fixed vectors in $\mathcal{R}^4$ for (1) the column space of A [25%] and (2) the row space A [25%]. The two displayed bases **must** consist of columns of A and rows of A, respectively.

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 6 & 6 & 4 \\ 0 & -3 & -3 & -2 \\ 0 & -1 & -2 & -1 \end{pmatrix}$$

If you did (a), (b) and (c), then 100% has been marked – go on to Ch5. Otherwise, unmark one of (a), (b) or (c), then complete (d). Only three problems will be graded.

☐ [25%] Ch4(d): Define S to be the set of all vectors $\mathbf{x}$ in $\mathcal{R}^4$ such that $x_1 = x_3 + x_4$ and $x_2 = 3x_4$. Prove or disprove that S is a subspace of $\mathcal{R}^4$.

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Staple this page to the top of all Ch4 work. Submit one package per chapter.

Ch 4 (a) (1) Rank Test : v_1, v_2, v_3 independent $\Leftrightarrow$ $\operatorname{rank}(\operatorname{aug}(v_1, v_2, v_3)) = 3$

(2) Det Test : v_1, v_2, v_3 independent $\Leftrightarrow$ $\det(\operatorname{aug}(v_1, v_2, v_3)) \neq 0$.

Apply : Only rank test applies.

$$A = \begin{pmatrix} 1 & -1 & -1 \\ 2 & 0 & 2 \\ 1 & 3 & 7 \\ 0 & 1 & 2 \end{pmatrix} \quad \operatorname{rref}(A) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{rank} = 2 \quad \boxed{\text{Dependent}}$$

Ch 4 (b) $\boxed{\text{NO}}$ $\operatorname{rref}(B) = \operatorname{rref}(C)$ does not imply the bases are equivalent.

Explain : $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}$ both have $\operatorname{rref} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$

But $\operatorname{col}(C, 2)$ is not a linear combination of the columns of B.

Ch 4 (c) In the frame sequence $A \to \operatorname{rref}(A)$ appears $\begin{pmatrix} 0 & 1 & 2 & 1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

So pivots of A = 2, 3.

In frame sequence $A^T \to \operatorname{rref}(A^T)$ appears $\begin{pmatrix} 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

So pivots of A^T = 2, 4

Rowspace = $\operatorname{span}\{\operatorname{row}(A,2), \operatorname{row}(A,4)\}$

Col Space = $\operatorname{span}\{\operatorname{col}(A,2), \operatorname{col}(A,3)\}$

Ch 4 (d) Let $A = \begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Then $A\vec{x} = \vec{0}$ is equivalent to

the system $\begin{cases} x_1 - x_3 - x_4 = 0 \\ x_2 - 3x_4 = 0 \end{cases}$. By the kernel theorem,

$S = \{\vec{x} : A\vec{x} = \vec{0}\}$ is a subspace of R^4.

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Ch5. (Linear Equations of Higher Order)

☐ [10%] Ch5(a): Find the general solution of the differential equation

$$y'' + \frac{50}{7}y' + y = 0.$$

☐ [20%] Ch5(b): Find the homogeneous differential equation general solution, given characteristic equation

$$(r^2 - 5r)^3(r^4 + 25r^2)(5r^2 + 26r + 5)^2 = 0.$$

☐ [10%] Ch5(c): Given a damped spring-mass system $mx''(t) + cx'(t) + kx(t) = 0$ with $m = 6$, $c = 37$ and $k = 6$, classify the answer as over-damped, critically damped or under-damped. Please, **do not solve** the differential equation!

☐ [40%] Ch5(d): Assume a seventh order constant-coefficient differential equation has characteristic equation $r(r^2 + 4)(r^2 + 16)^2 = 0$. Suppose the right side of the differential equation is

$$f(x) = x(x + 2\cos 2x) + \sin 4x$$

Determine the **corrected** trial solution for y_p according to the method of undetermined coefficients. **Do not evaluate** the undetermined coefficients!

☐ [20%] Ch5(e): Find the steady-state periodic solution for the equation

$$x'' + 4x' + 20x = \cos(5t).$$

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ch5 (a) $7r^2+50r+7=0$

$(7r+1)(r+7)=0$

atoms = $e^{-x/7}, e^{-7x}$ $\quad y = c_1 e^{-x/7} + c_2 e^{-7x}$

ch5 (b) $r^3(r-5)^3 r^2 (r^2+25)(5r+1)^2(r+5)^2 = 0$

atoms = $1, x, x^2, x^3, x^4, e^{5x}, xe^{5x}, x^2e^{5x}, \cos 5x, \sin 5x,$
$e^{-x/5}, xe^{-x/5}, e^{-5x}, xe^{-5x}$

y = linear combination of the 14 atoms using $c_1 \to c_{14}$

ch5 (c) $6r^2+37r+6=0$

$(6r+1)(r+6)=0$ $\quad$ over-damped

2 real roots

ch5 (d) atoms of f = $1, x, x^2, \cos 2x, \sin 2x, x\cos 2x, x\sin 2x,$
$\cos 4x, \sin 4x$

char eq roots $= 0, 2i, -2i, 4i, -4i, 4i, -4i$

$$y = x(d_1 + d_2 x + d_3 x^2)$$
$$+ x^2(d_4 \cos 4x + d_5 \sin 4x)$$
$$+ x(d_6 \cos 2x + d_7 \sin 2x + d_8 x\cos 2x + d_9 x \sin 2x)$$

ch5 (e) No fixup rule for $x = d_1 \cos 5t + d_2 \sin 5t$, because the char eq roots are $-2 \pm 4i$.

$$\begin{cases} -5d_1 + 20d_2 = 1 \\ -20d_1 - 5d_2 = 0 \end{cases} \quad \text{Solving,} \quad \begin{cases} d_1 = \frac{-1}{85} \\ d_2 = \frac{4}{85} \end{cases}$$

$$x_{ss} = -\frac{1}{85}\cos 5t + \frac{4}{85}\sin 5t$$

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Ch6. (Eigenvalues and Eigenvectors)

☐ [25%] Ch6(a): Find the eigenvalues of the matrix

$$A = \begin{pmatrix} 0 & 4 & -1 & 0 \\ 4 & 0 & -2 & 1 \\ 0 & 0 & -5/3 & 2/3 \\ 0 & 0 & 2/3 & -5/3 \end{pmatrix}.$$

To save time, **do not** find eigenvectors!

☐ [25%] Ch6(b): Let $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Prove or disprove: Fourier's model holds for A.

☐ [25%] Ch6(c): Find a 2×2 matrix A with eigenpairs

$$\left(1, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right), \quad \left(-2, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right).$$

☐ [25%] Ch6(d): The matrix A below has eigenvalue package

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -6 \end{pmatrix}$$

State whether A is diagonalizable, and if it is, then display the matrix package P of eigenvectors.

$$A = \begin{pmatrix} 1 & 4 & 3 \\ 2 & -2 & -2 \\ 3 & -4 & 1 \end{pmatrix}$$

If you did (a), (b), (c) and (d), then 100% has been marked – go on to Ch7. Otherwise, unmark one of (a), (b), (c) or (d), then complete (e). Only four problems will be graded.

☐ [25%] Ch6(e): Let 3×3 matrices A and B be related by $AP = PB$ for some invertible matrix P. Prove or disprove: the roots of the characteristic equations of A and B are identical.

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Staple this page to the top of all Ch6 work. Submit one package per chapter.

ch6 (a) $(\lambda^2-16)\left(\left(\frac{5}{3}+\lambda\right)^2-\frac{4}{9}\right)=0$ $\lambda=$ $\boxed{4,-4,\frac{-7}{3},-1}$

ch6 (b) Fourier's model does not hold because A is not diagonalizable. All $\lambda=0$ because A is triangular. $B=A-\lambda I=A$ for $\lambda=0$. Already $B=\text{rref}(B)$. Only eigenvectors are $t_1\begin{pmatrix}1\\0\\0\end{pmatrix}$. Because there are less than 3 eigenpairs, then A is not diagonalizable and Fourier's model does not hold.

ch6 (c) Let $D=\begin{pmatrix}1&0\\0&-2\end{pmatrix}$, $P=\begin{pmatrix}1&1\\2&1\end{pmatrix}$. $AP=PD \Longrightarrow$

$$A=PDP^{-1}$$
$$=\begin{pmatrix}1&1\\2&1\end{pmatrix}\begin{pmatrix}1&0\\0&-2\end{pmatrix}\begin{pmatrix}1&-1\\-2&1\end{pmatrix}\frac{1}{-1}$$
$$=\begin{pmatrix}1&-2\\2&-2\end{pmatrix}\begin{pmatrix}1&-1\\-2&1\end{pmatrix}(-1)$$
$$\boxed{A=\begin{pmatrix}-5&3\\-6&4\end{pmatrix}}$$

ch6 (d) A is diagonalizable because it has 3 distinct eigenvalues $2,4,-6$, the diagonal elements of D.

$P=\begin{pmatrix}1&1&-1\\1&0&1\\-1&1&1\end{pmatrix}$ by 3 frame sequences.

[one page computation expected]

ch6 (e) Theorem A,B 3×3 and $AP=PB$ with P invertible $\Longrightarrow$ $\det(A-\lambda I)=\det(B-\lambda I)$

proof: $\det(A-\lambda I)=\det(PBP^{-1}-\lambda I)$
$$=\det(P(B-\lambda I)P^{-1})$$
$$=\det P\ \det(B-\lambda I)\ \det P^{-1}$$
$$=\det PP^{-1}\ \det(B-\lambda I)$$
$$=\det(B-\lambda I)$$

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Ch7. (Linear Systems of Differential Equations)

☐ [25%] Ch7(a): Apply the eigenanalysis method to solve the system $\mathbf{x}' = A\mathbf{x}$, given A has one eigenvalue $\lambda = 4$ and

$$A = \begin{pmatrix} -4 & -6 & 8 \\ 2 & -2 & -2 \\ 2 & -6 & 2 \end{pmatrix}.$$

☐ [25%] Ch7(b): Give an example of a 2×2 real matrix A for which Fourier's model is not valid. Then display the general solution $\mathbf{x}(t)$ of $\mathbf{x}' = A\mathbf{x}$.

☐ [25%] Ch7(c): Solve for $y(t)$ in the system below.

$$\begin{aligned} x' &= x + 5y, \\ y' &= -x + y. \end{aligned}$$

☐ [25%] Ch7(d): Let A be a 2×2 real matrix and assume Fourier's model is valid for A. Display the general solution $\mathbf{x}(t)$ for $\mathbf{x}' = A\mathbf{x}$ in terms of the ingredients of Fourier's model.

If you solved (a), (b), (c) and (d), then you have marked 100%. If so, then go on to Ch10, otherwise, continue here. Only 4 parts will be graded.

☐ [25%] Ch7(e): Consider a 3×3 system $\mathbf{x}' = A\mathbf{x}$. Assume A has an eigenvalue $\lambda = -0.001$ with corresponding eigenvectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Find a nonzero solution of the differential equation with limit zero at infinity.

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Staple this page to the top of all Ch7 work. Submit one package per chapter.

ch 7 (a) By cofactor expansion

$$\det(A-\lambda I) = -\lambda^3 - 4\lambda^2 + 20\lambda + 48$$

$\lambda = 4$ is a root. Divide

$$\begin{array}{r l} & -\lambda^2 - 8\lambda - 12 \\ \lambda - 4 & \overline{\big)\, -\lambda^3 - 4\lambda^2 + 20\lambda + 48} \\ & -\lambda^3 + 4\lambda^2 \\ & \overline{\quad -8\lambda^2 + 20\lambda + 48} \\ & \quad -8\lambda^2 + 32\lambda \\ & \overline{\qquad -12\lambda + 48} \\ & \qquad -12\lambda + 48 \\ & \overline{\qquad\qquad 0} \end{array}$$

Eigen pairs =

$$\left(4, \begin{pmatrix}1\\0\\1\end{pmatrix}\right), \left(-6, \begin{pmatrix}-1\\1\\1\end{pmatrix}\right), \left(-2, \begin{pmatrix}1\\1\\1\end{pmatrix}\right)$$

$$\det(A-\lambda I) = (\lambda-4)(\lambda^2+8\lambda+12)(-1)$$
$$= (\lambda-4)(\lambda+6)(\lambda+2)(-1)$$

$$\vec{x}(t) = c_1 e^{4t}\begin{pmatrix}1\\0\\1\end{pmatrix} + c_2 e^{-6t}\begin{pmatrix}-1\\1\\1\end{pmatrix} + c_3 e^{-2t}\begin{pmatrix}1\\1\\1\end{pmatrix}$$

ch 7 (b) $A = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$ $\quad \begin{cases} x_1' = x_2 \\ x_2' = 0 \end{cases}$ $\quad \boxed{\begin{array}{l} x_1 = c_2 t + c_1 \\ x_2 = c_2 \end{array}}$

ch 7 (c) $r^2 - 2r + 6 = 0$ or $(r-1)^2 + 5 = 0$

$$y(t) = c_1 e^t \cos\sqrt{5}\,t + c_2 e^t \sin\sqrt{5}\,t$$

ch 7 (d) $A(c_1\vec{v}_1 + c_2\vec{v}_2) = c_1\lambda_1\vec{v}_1 + c_2\lambda_2\vec{v}_2 \implies$

$$x(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2$$

ch 7 (e) $\vec{x}(t) = e^{-0.001t}\vec{v}_1$ is a solution with limit zero at $t = \infty$.

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Ch10. (Laplace Transform Methods)

It is assumed that you have memorized the basic 4-item Laplace integral table and know the 6 basic rules for Laplace integrals. No other tables or theory are required to solve the problems below. If you don't know a table entry, then leave the expression unevaluated for partial credit.

☐ [25%] Ch10(a): Apply Laplace's method to the system to find a 2×2 system for $\mathcal{L}(x)$, $\mathcal{L}(y)$. To save time, **do not solve** the system and don't find $x(t)$ or $y(t)$!

$$\begin{aligned} &x'' = y, \\ &y'' = 16x, \\ &x(0) = 0, \quad x'(0) = 1, \\ &y(0) = 0, \quad y'(0) = 0. \end{aligned}$$

☐ [25%] Ch10(b): Solve for $f(t)$, given

$$\mathcal{L}(f(t)) = \left(\frac{d}{ds}\left(\mathcal{L}(te^t \cos t)\right)\right)\Big|_{s \to (s+2)} + \frac{s+2}{(s-1)^3} + \mathcal{L}\left(e^{-2t}\left(t \sin 2t\right)\right)$$

☐ [25%] Ch10(c): Find $f(t)$ by partial fraction methods, given

$$\mathcal{L}(f(t)) = \frac{2+3s}{s^2+2s} + \frac{s^2-3}{(s-1)^2(s-2)}.$$

☐ [25%] Ch10(d): Apply Laplace's method to find a formula for $\mathcal{L}(x(t))$. **Do not** solve for $x(t)$! Document steps by reference to tables and rules.

$$x''' + 4x'' = e^t(5t + e^t \sin 2t), \quad x(0) = 1, \quad x'(0) = x''(0) = 0.$$

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ch10 (a)

$$-x'(0)+s\mathcal{L}(x')=\mathcal{L}(y)$$
$$-1+s(-x(0)+s\mathcal{L}(x))=\mathcal{L}(y)$$
① $$s^2\mathcal{L}(x)+(-1)\mathcal{L}(y)=1$$
$$-y'(0)+s\mathcal{L}(y')=16\,\mathcal{L}(x)$$
$$s^2\mathcal{L}(y)=16\,\mathcal{L}(x)$$
② $$16\,\mathcal{L}(x)+(-s^2)\mathcal{L}(y)=0$$

$$\boxed{\begin{cases} s^2\mathcal{L}(x)-\mathcal{L}(y)=1 \\ 16\,\mathcal{L}(x)-s^2\mathcal{L}(y)=0 \end{cases}}$$

ch10 (b)

$$\mathcal{L}(f)=\left(\mathcal{L}(-t^2e^t\cos t)\right)\Big|_{s\to s+2}+\frac{s+3}{s^3}\Big|_{s\to s-1}+\mathcal{L}(te^{-2t}\sin 2t)$$
$$=\mathcal{L}(-t^2e^{-t}\cos t)+\left(\frac{1}{s^2}+\frac{3}{s^3}\right)\Big|_{s\to s-1}+\mathcal{L}(te^{-2t}\sin 2t)$$
$$=\mathcal{L}(-t^2e^{-t}\cos t+te^t+\tfrac{3}{2}t^2e^t+te^{-2t}\sin 2t)$$
$$f=-t^2e^{-t}\cos t+(t+\tfrac{3}{2}t^2)e^t+te^{-2t}\sin 2t$$

ch10 (c)

$$\mathcal{L}(f)=\frac{2+3s}{s(s+2)}+\frac{s^2-3}{(s-1)^2(s-2)}$$
$$=\frac{a}{s}+\frac{b}{s+2}+\frac{c}{s-1}+\frac{d}{(s-1)^2}+\frac{f}{s-2}$$
$$=\mathcal{L}(a+be^{-2t}+ce^t+dte^t+fe^{2t})$$
$$f=a+be^{-2t}+ce^t+dte^t+fe^{2t}$$
$$f=1+2e^{-2t}+0+2te^t+e^{2t}$$

ch10 (d)

$$\mathcal{L}(x''')+4\mathcal{L}(x'')=\mathcal{L}(e^t(5t+e^t\sin 2t))$$
$$-x''(0)+s\mathcal{L}(x'')+4(-x'(0)+s\mathcal{L}(x'))= \quad ''$$
$$s(-x'(0)+s\mathcal{L}(x'))+4s(-x(0)+s\mathcal{L}(x))= \quad ''$$
$$s^2(-x(0)+s\mathcal{L}(x))-4s+4s^2\mathcal{L}(x)= \quad ''$$
$$-s^2+s^3\mathcal{L}(x)-4s+4s^2\mathcal{L}(x)=$$

$$\mathcal{L}(x)=\frac{s^2+4s+\mathcal{L}(e^t(5t+e^t\sin 2t))}{s^3+4s^2}$$
$$=\frac{s^2+4s+\left(\frac{5}{s^2}+\frac{2}{(s-1)^2+4}\right)\Big|_{s\to s-1}}{s^3+4s^2}$$
$$=\frac{s^2+4s+5/(s-1)^2+\frac{2}{(s-2)^2+4}}{s^3+4s^2}$$