Name. ______________________

Differential Equations 5420
Midterm Exam 1, Spring 2003
Due Date: January 31, 2003

Instructions. The four problems below are take-home, due on the date above. Answer checks are expected. If `maple` assist is used, then please attach the `maple` output.

1. **(Matrix Exponential)** Prove that the matrix series $\sum_{n=0}^{\infty} A^n/n!$ converges. Do it by showing that each element of the partial sum matrix $S_N = I + \cdots + A^N/N!$ corresponds to a Cauchy sequence of real numbers.

2. **(Exponential identities)** Prove that: (1) $AB = BA$ implies $e^{A+B} = e^A e^B = e^B e^A$, (2) $A = \textbf{diag}(A1, A2, A3)$ implies $e^A = \textbf{diag}(e^{A1}, e^{A2}, e^{A3})$, (3) $e^N = I + \cdots + N^k/k!$ provided $N^{k+1} = 0$.

3. **(Eigenanalysis)** Find the eigenvalues, eigenvectors and generalized eigenvector chains for the matrix

$$A = \begin{bmatrix} 12 & 25 & 0 & 0 & 0 & 0 & 0 \\ -5 & -8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

   Then solve $X' = AX$.

4. **(Solving $X' = AX$)** Give an example of a system $X' = AX$ with A real 4×4 having solution components involving only $\sin 2t$, $\cos 2t$, $t \sin 2t$, $t \cos 2t$. Please display A, solve the system $X' = AX$ and verify the solution.

5. **(Finite dimensional spectral theory)** Using Hirsch-Smale as a reference, write a summary of finite dimensional spectral theory, as it applies to the problem $X' = AX$ with A $n \times n$ real. Describe in detail how to form the real matrix P of generalized eigenvectors from the chains of generalized eigenpairs. Include one illustration which computes from A the chains, P and the real Jordan form, followed by solving $X' = AX$.